CHAPTER 1

Combinatorics

1.1. Introduction

The first basic principle is to multiply.

Example 1.1: Suppose we have 4 shirts of 4 different colors and 3 pants of different colors. How many possibilities are there? For each shirt there are 3 possibilities, so altogether there are \(4 \times 3 = 12\) possibilities.

Example 1.2: How many license plates of 3 letters followed by 3 numbers are possible? 
Answer. \((26)^3(10)^3\), because there are 26 possibilities for the first place, 26 for the second, 26 for the third, 10 for the fourth, 10 for the fifth, and 10 for the sixth. We multiply.

How many ways can one arrange \(a, b, c\)? One can have 
\[abc, \ acb, \ bca, \ cab, \ cba.\]
There are 3 possibilities for the first position. Once we have chosen the first position, there are 2 possibilities for the second position, and once we have chosen the first two possibilities, there is only 1 choice left for the third. So there are \(3 \times 2 \times 1 = 6 = 3!\) arrangements. In general, if there are \(n\) letters, there are \(n!\) possibilities.

Example 1.3: What is the number of possible batting orders with 9 players?
Answer. \(9! = 362880\)

Example 1.4: How many ways can one arrange 4 math books, 3 chemistry books, 2 physics books, and 1 biology book on a bookshelf so that all the math books are together, all the chemistry books are together, and all the physics books are together.

Answer. \(4! \cdot (4! \cdot 3! \cdot 2! \cdot 1!) = 6912\). We can arrange the math books in \(4!\) ways, the chemistry books in \(3!\) ways, the physics books in \(2!\) ways, and the biology book in \(1! = 1\) way. But we also have to decide which set of books go on the left, which next, and so on. That is the same as the number of ways of arranging the letters \(M, C, P, B\), and there are \(4!\) ways of doing that.

Example 1.5: How many ways can one arrange the letters \(a, a, b, c\)? Let us label them \(A,a,b,c\). There are \(4!\), or 24, ways to arrange these letters. But we have repeats: we could
have $Aa$ or $aA$. So we have a repeat for each possibility, and so the answer should be $4!/2! = 12$.

If there were 3 $a$’s, 4 $b$’s, and 2 $c$’s, we would have

$$\frac{9!}{3!4!2!} = 1260.$$  

What we just did was called the **number of permutations**.

Now let us look at what are known as **combinations**. How many ways can we choose 3 letters out of 5? If the letters are $a,b,c,d,e$ and order matters, then there would be 5 for the first position, 4 for the second, and 3 for the third, for a total of $5 \times 4 \times 3$. But suppose the letters selected were $a,b,c$. If order doesn’t matter, we will have the letters $a,b,c$ 6 times, because there are $3!$ ways of arranging 3 letters. The same is true for any choice of three letters. So we should have $5 \times 4 \times 3 / 3!$. We can rewrite this as

$$\frac{5 \cdot 4 \cdot 3}{3!} = \frac{5!}{3!2!} = 10$$

This is often written $\binom{5}{3}$, read “5 choose 3.” Sometimes this is written $C_{5,3}$ or $5C_3$. More generally,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$  

**Example 1.6:** How many ways can one choose a committee of 3 out of 10 people?

**Answer.** $\binom{10}{3} = 120$.

**Example 1.7:** Suppose there are 8 men and 8 women. How many ways can we choose a committee that has 2 men and 2 women?

**Answer.** We can choose 2 men in $\binom{8}{2}$ ways and 2 women in $\binom{8}{2}$ ways. The number of committees is then the product: $\binom{8}{2} \cdot \binom{8}{2} = 784$.

**Example 1.8:** Suppose one has 9 people and one wants to divide them into one committee of 3, one of 4, and a last of 2. There are $\binom{9}{3}$ ways of choosing the first committee. Once that is done, there are 6 people left and there are $\binom{6}{4}$ ways of choosing the second committee. Once that is done, the remainder must go in the third committee. So the answer is

$$\frac{9! \cdot 6!}{3!6!4!2!} = \frac{9!}{3!4!2!}.$$
1.1. INTRODUCTION

In general, to divide \( n \) objects into one group of \( n_1 \), one group of \( n_2 \), ..., and a \( k \)th group of \( n_k \), where \( n = n_1 + \cdots + n_k \), the answer is

\[
\frac{n!}{n_1!n_2!\cdots n_k!}.
\]

These are known as multinomial coefficients.

Another example: suppose we have 4 Americans and 6 Canadians. (a) How many ways can we arrange them in a line? (b) How many ways if all the Americans have to stand together? (c) How many ways if not all the Americans are together? (d) Suppose you want to choose a committee of 3, which will be all Americans or all Canadians. How many ways can this be done? (e) How many ways for a committee of 3 that is not all Americans or all Canadians?

Answer. (a) This is just \( 10! \) (b) Consider the Americans as a group and each Canadian as a group; this gives 7 groups, which can be arranged in \( 7! \) ways. Once we have these seven groups arranged, we can arrange the Americans within their group in \( 4! \) ways, so we get \( 4!7! \). (c) This is the answer to (a) minus the answer to (b): \( 10! - 4!7! \). (d) We can choose a committee of 3 Americans in \( \binom{4}{3} \) ways and a committee of 3 Canadians in \( \binom{6}{3} \) ways, so the answer is \( \binom{4}{3} + \binom{6}{3} \). (e) We can choose a committee of 3 out of 10 in \( \binom{10}{3} \) ways, so the answer is \( \binom{10}{3} - \binom{4}{3} - \binom{6}{3} \).

Finally, we consider three interrelated examples. First, suppose one has 8 o’s and 2 |’s. How many ways can one arrange these symbols in order? There are 10 spots, and we want to select 8 of them in which we place the o’s. So we have \( \binom{10}{8} \).

Next, suppose one has 8 indistinguishable balls. How many ways can one put them in 3 boxes? Let us make sequences of o’s and |’s; any such sequence that has | at each side, 2 other |’s, and 8 o’s represents a way of arranging balls into boxes. For example, if one has

\[
|\ o\ o\ |\ o\ o\ o\ |\ o\ o\ o\ |
\]

this would represent 2 balls in the first box, 3 in the second, and 3 in the third. Altogether there are \( 8 + 4 \) symbols, the first is a | as is the last. so there are 10 symbols that can be either | or o. Also, 8 of them must be o. How many ways out of 10 spaces can one pick 8 of them into which to put a o? We just did that: the answer is \( \binom{10}{8} \).

Now, to finish, suppose we have $8,000 to invest in 3 mutual funds. Each mutual fund required you to make investments in increments of $1,000. How many ways can we do this? This is the same as putting 8 indistinguishable balls in 3 boxes, and we know the answer is \( \binom{10}{8} \).
1.2. Further examples and explanations

1.2.1. Counting Principle revisited. We need a way to help us count faster rather than counting by hand one by one. We define the following counting principle.

**Fact.** (Basic Counting Principle) Suppose 2 experiments are to be performed. If one experiment can result in $m$ possibilities, and the second experiment can result in $n$ possibilities, then together there are $mn$ possibilities.

One can visualize the Basic Counting Principle by using the Box Method. In the Box Method, each box represents the number of possibilities in that experiment.

<table>
<thead>
<tr>
<th>Experiment 1</th>
<th>Experiment 2</th>
<th>Experiment 1 and 2 together</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>$m$</td>
<td>$mn$</td>
</tr>
</tbody>
</table>

**Example 1.9:** There are 20 teachers and 100 students in a school. How many ways can we pick a teacher and student of the year?

Answer. Use the box method: $20 \times 100 = 2000$.

**Fact.** The counting principle can be generalized to any number of experiments: for $k$ experiment we have $n_1 \cdots n_k$ possibilities.

**Example 1.10:** A college planning committee consists of 3 freshmen, 4 sophomores, 5 juniors, and 2 seniors. A subcommittee of 4 consists 1 person from each class. How many choices are possible?

Answer. Box Method gives $3 \times 4 \times 5 \times 2 = 120$.

**Example 1.11:** Recall that for 6-place license plates, with the first three places occupied by letters and the last three by numbers, we have $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10$ choices. Question: What if no repetition is allowed?

Answer. the Box Method again: $26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8$

**Example 1.12:** How many functions defined on $k$ points are possible if each functional value is either 0 or 1.

Answer. Box method on the $1, \ldots, k$ points gives us $2^k$ possible functions. This is the generalized counting principle with $n_1 = n_2 = \ldots = n_k = 2$. 
1.2.2. **Permutations.** Recall how many different ordered arrangements of the letters $a, b, c$ are possible:

- $abc, acb, bac, bca, cab, cba$, and each arrangement is a permutation.
- We also can use the Box Method to figure this out: $3 \cdot 2 \cdot 1 = 6$.

**FACT.** With $n$ objects, there are

$$n \ (n - 1) \cdots 3 \cdot 2 \cdot 1 = n!$$

different permutations of the $n$ objects.

(*) Note that order matters when it comes to permutations.

**Example 1.13:** What is the number of possible batting order with 9 players?

**Answer.** $9!$ by the Box Method or permutations.

**Example 1.14:** How many ways can one arrange 5 math books, 6 chemistry books, 7 physics books, and 8 biology books on a bookshelf so that all the math books are together, all the chemistry books are together, and all the physics books are together.

**Answer.** We can arrange the math books in $5!$ ways, the chemistry in $6!$ ways, the physics in $7!$ ways, and biology books in $8!$ ways. We also have to decide which set of books go on the left, which next, and so on. That is the same as the number of ways of arranging the letters $M, C, P, B$, and there are $4!$ ways of doing that. So the total is $4! \cdot (5! \cdot 6! \cdot 7! \cdot 8!)$ ways.

Now consider a couple of examples with **Repetitions.**

**Example 1.15:** How many ways can one arrange the letters $a, a, b, b, c, c$? Let us first re-label them $A, a, B, b, C, c$. Then there are $6! = 720$, ways to arrange these letters. But we have repeats: we could have $Aa$ or $aA$. So we have a repeat for each possibility ans (so we have to divide!).

**Answer.** $6! / (2!)^3 = 60$.

**Example 1.16:** How many different letter arrangements can be formed from the word PEPPER?

**Answer.** There $3$ $P$’s $2$ $E$’s and one $R$. So $6! / 3! 2! 1! = 60$.

**Example 1.17:** Suppose there are 4 Czech tennis players, 4 U.S. players, and 3 Russian players, in how many ways could they be arranged, if we don’t distinguish players from the same country?

**Answer.** $11! / 4! 4! 3!$.

**FACT.** There are

$$\frac{n!}{n_1! \cdots n_r!}$$

different permutations of $n$ objects of which $n_1$ are alike, $n_2$ are alike, ..., $n_r$ are alike.
1.2.3. Combinations. We are often interested in selecting \( r \) objects from a total of \( n \) objects and the order of these objects does not matter.

**Fact.** If \( r \leq n \), then

\[
{n \choose r} = \frac{n!}{(n-r)!r!}
\]

called \( n \) choose \( r \), represents the number of possible combinations of objects taken \( r \) at a time from \( n \) objects.

(*) The order **DOES NOT** matter for combinations.

Recall in Permutations order did matter.

**Example 1.18:** How many ways can one choose a committee of 3 out of 10 people?

Answer. \( \binom{10}{3} = \frac{10!}{3!7!} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2} = 10 \cdot 3 \cdot 4 = 120 \).

**Example 1.19:** Suppose there are 9 men and 8 women. How many ways can we choose a committee that has 2 men and 3 women?

Answer. We can choose 2 men in \( \binom{9}{2} \) ways and 3 women in \( \binom{8}{3} \) ways. The number of committees is then the product \( \binom{9}{2} \cdot \binom{8}{3} \).

**Example 1.20:** Suppose somebody has \( n \) friends, of whom \( k \) are to be invited to a meeting.

Answer.

a How many choices exist if 2 of the friends will not attend together?

- Box it: [none] + [one of them] [others]
- \( \binom{n-2}{k} + \binom{2}{1} \cdot \binom{n-2}{k-1} \) (recall that when we have OR, use +)

b How many choices exist if 2 of the friends will only attend together?

- Box it: [none] + [with both]
- \( \binom{n-2}{k} + 1 \cdot 1 \cdot \binom{n-2}{k-2} \)

The value of \( \binom{n}{r} \) are called binomials coefficients because of their prominence in the binomial theorem.

**Theorem.** *(The Binomial Theorem)*

\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.
\]
Proof. To see this, the left hand side is \((x + y)(x + y) \cdots (x + y)\). This will be the sum of \(2^n\) terms, and each term will have \(n\) factors. How many terms have \(k\) \(x\)'s and \(n-k\) \(y\)'s? This is the same as asking in a sequence of \(n\) positions, how many ways can one choose \(k\) of them in which to put \(x\)'s? (Box it) The answer is \(\binom{n}{k}\), so the coefficient of \(x^k y^{n-k}\) should be \(\binom{n}{k}\).

Example 1.21: Using Combinatorics: Let’s prove

\[
\binom{10}{4} = \binom{9}{3} + \binom{9}{4}
\]

with no algebra:

Answer. The left hand side (LHS) represents the number of committees having 4 people out of the 10. Let’s interpret the right hand side (RHS). Let’s say Tom Brady will be in one of these committees and he’s special, so we want to know when he’ll be there or not. When he’s there, then there are \(1\cdot\binom{9}{3}\) number of ways that contain Tom Brady while \(\binom{9}{4}\) is the number of committees that do not contain Tom Brady and contain 4 out of the remaining people. Adding it up gives us the number of committees having 4 people out of the 10.

Example 1.22: The more general equation is

\[
\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}
\]

Example 1.23: Expand \((x + y)^3\).

Answer. \((x + y)^3 = y^3 + 3xy^2 + 3x^2y + x^3\).
1.2.4. Multinomial Coefficients.

Example 1.24: Suppose one has 9 people and wants to divide them into one committee of 3, one of 4, and a last of 2. How many different ways are there?

Answer. (Box it) There are \( \binom{9}{3} \) ways of choosing the first committee. Once that is done, there are 6 people left and there are \( \binom{6}{4} \) ways of choosing the second committee. Once that is done, the remainder must go in the third committee. So there is 1 to choose that. So the answer is

\[
\frac{9!}{3!6!} = \frac{9!}{3!4!2!}
\]

In general: if we are to divide \( n \) objects into a group of \( n_1 \), a group of \( n_2, \ldots \) and a \( k \)th group of \( n_k \), where \( n = n_1 + \cdots + n_k \), then the answer can be given in \( \binom{n!}{n_1!n_2!\ldots n_k!} \) ways. These are known as multinomial coefficients. We write them as

\[
\binom{n}{n_1,n_2,\ldots,n_k} = \frac{n!}{n_1!n_2!\ldots n_k!}
\]

Example 1.25: Suppose we are to assign 10 police officers: 6 patrols, 2 in station, 2 in schools. Then there are \( \frac{10!}{6!2!2!} \) different assignments.

Example 1.26: There are 10 flags: 5 Blue one, 3 red one, and 2 yellow. These flags are indistinguishable, except for their color. How many different ways can we order them on a flag pole?

Answer. \( \frac{10!}{5!3!2!} \).

Example 1.27: Suppose one has \( n \) indistinguishable balls. How many ways can one put them in \( k \) boxes, assuming \( n > k \)?

Solution 1: Let us make sequences of \( o \)'s and \( | \)'s; any such sequence that has \( | \) at each side, \( k - 1 \) other \( | \)'s, and \( n \) \( o \)'s represents a way of arranging balls into boxes. For example, one may have

\[ | o o o | o o o | \]

if \( n = 8 \) and \( k = 3 \). How many different ways can we arrange this, if we have to start with \( | \) and end with \( | \)? In between, we are only arranging \( n + k - 1 \) symbols, of which only \( n \) are \( o \)'s. So the question is: how many ways out of \( n + k - 1 \) spaces can one pick \( n \) of them into which to put an \( o \)? Answer: \( \binom{n + k - 1}{n} \).

Solution 2: Look at spaces between that have a \( | \). There are \( k - 1 \) spaces, and so the answer is 

\[
\binom{n + k - 1}{k - 1} = \binom{n + k - 1}{n}
\]
1.3. Exercises

Exercise 1.1: Suppose a license plate must consist of 7 numbers or letters. How many license plates are there if

(A) there can only be letters?
(B) the first three places are numbers and the last four are letters?
(C) the first three places are numbers and the last four are letters, but there can not be any repetitions in the same license plate?

Exercise 1.2: A school of 50 students has awards for the top math, English, history and science student in the school

(A) How many ways can these awards be given if each student can only win one award?
(B) How many ways can these awards be given if students can win multiple awards?

Exercise 1.3: A password can be made up of any 4 digit combination.

(A) How many different passwords are possible?
(B) How many are possible if all the digits are odd?
(C) How many can be made in which all digits are different or all digits are the same?

Exercise 1.4: There is a school class of 25 people made up of 11 guys and 14 girls.

(A) How many ways are there to make a committee of 5 people?
(B) How many ways are there to pick a committee of all girls?
(C) How many ways are there to pick a committee of 3 girls and 2 guys?

Exercise 1.5: If a student council contains 10 people, how many ways are there to elect a president, a vice president, and a 3 person prom committee from the group of 10 students?

Exercise 1.6: Suppose you are organizing your textbooks on a book shelf. You have three chemistry books, 5 math books, 2 history books and 3 English books.

(A) How many ways can you order the textbooks if you must have math books first, English books second, chemistry third, and history fourth?
(B) How many ways can you order the books if each subject must be ordered together?

Exercise 1.7: If you buy a Powerball lottery ticket, you can choose 5 numbers between 1 and 59 (picked on white balls) and one number between 1 and 35 (picked on a red ball). How many ways can you

(A) win the jackpot (guess all the numbers correctly)?
(B) match all the white balls but not the red ball?
(C) match exactly 3 white balls and the red ball?
(D) match at least 3 white balls and the red ball?
**Exercise 1.8:** A couple wants to invite their friends to be in their wedding party. The groom has 8 possible groomsmen and the bride has 11 possible bridesmaids. The wedding party will consist of 5 groomsmen and 5 bridesmaids.

(A) How many wedding party’s are possible?
(B) Suppose that two of the possible groomsmen are feuding and will only accept an invitation if the other one is not going. How many wedding party’s are possible?
(C) Suppose that two of the possible bridesmaids are feuding and will only accept an invitation if the other one is not going. How many wedding party’s are possible?
(D) Suppose that one possible groomsmen and one possible bridesmaid refuse to serve together. How many wedding party’s are possible?

**Exercise 1.9:** There are 52 cards in a standard deck of playing cards. The poker hand consists of five cards. How many poker hands are there?

**Exercise 1.10:** There are 30 people in a communications class. Each student must interview one another for a class project. How many total interviews will there be?

**Exercise 1.11:** Suppose a college basketball tournament consists of 64 teams playing head to head in a knockout style tournament. There are 6 rounds, the round of 64, round of 32, round of 16, round of 8, the final four teams, and the finals. Suppose you are filling out a bracket, such as this, which specifies which teams will win each game in each round.

![Bracket Diagram]

How many possible brackets can you make?

**Exercise 1.12:** We need to choose a group of 3 women and 3 men out of 5 women and 6 men. In how many ways can we do it if 2 of the men refuse to be chosen together?

**Exercise 1.13:** Find the coefficient in front of $x^4$ in the expansion of $(2x^2 + 3y)^4$.

**Exercise 1.14:** In how many ways can you choose 2 or less (maybe none!) toppings for your ice-cream sundae if 6 different toppings are available? (You can use combinations here, but you do not have to.) Next, try to find a general formula to compute in how many ways you can choose $k$ or less toppings if $n$ different toppings are available.
1.4. Selected solutions

Solution to Exercise 1.1(A): $26^7$
Solution to Exercise 1.1(B): $10^3 \cdot 26^4$
Solution to Exercise 1.1(C): $10 \cdot 9 \cdot 8 \cdot 26 \cdot 25 \cdot 24 \cdot 23$

Solution to Exercise 1.2(A): $50 \cdot 49 \cdot 48 \cdot 47$
Solution to Exercise 1.2(B): $50^4$

Solution to Exercise 1.3(A): $10^4$
Solution to Exercise 1.3(B): $5^4$
Solution to Exercise 1.3(C): $10 \cdot 9 \cdot 8 \cdot 7 + 10$

Solution to Exercise 1.4(A): $\binom{25}{5}$
Solution to Exercise 1.4(B): $\binom{14}{5}$
Solution to Exercise 1.4(C): $\binom{14}{3} \cdot \binom{11}{2}$

Solution to Exercise 1.5: $10 \cdot 9 \cdot \binom{8}{3}$

Solution to Exercise 1.6(A): $5!3!3!2!$
Solution to Exercise 1.6(B): $4! \cdot (5!3!3!2!)$

Solution to Exercise 1.7(A): $1$
Solution to Exercise 1.7(B): $1 \cdot 34$
Solution to Exercise 1.7(C): $\binom{5}{3} \cdot \binom{54}{2} \cdot \binom{1}{1}$
Solution to Exercise 1.7(D): $\binom{5}{3} \cdot \binom{54}{2} \cdot \binom{1}{1} + \binom{5}{4} \cdot \binom{54}{1} \cdot \binom{1}{1} + 1$

Solution to Exercise 1.8(A): $\binom{8}{5} \cdot \binom{11}{5}$
Solution to Exercise 1.8(B): $\binom{6}{5} \cdot \binom{11}{5} + \binom{2}{1} \cdot \binom{6}{4} \cdot \binom{11}{5}$
Solution to Exercise 1.8(C): $\binom{8}{5} \cdot \binom{9}{5} + \binom{8}{5} \cdot \binom{2}{1} \cdot \binom{9}{4}$
Solution to Exercise 1.8(D): $\binom{7}{5} \cdot \binom{10}{5} + 1 \cdot \binom{7}{4} \cdot \binom{10}{5} + \binom{7}{5} \cdot 1 \cdot \binom{10}{4}$
Solution to Exercise 1.9: \[ \binom{52}{5} \]

Solution to Exercise 1.10: \[ \binom{30}{2} \]

Solution to Exercise 1.11: First notice that the 64 teams play 63 total games: 32 games in the first round, 16 in the second round, 8 in the 3rd round, 4 in the regional finals, 2 in the final four, and then the national championship game. That is, 32 + 16 + 8 + 4 + 2 + 1 = 63. Since there are 63 games to be played, and you have two choices at each stage in your bracket, there are \(2^{63}\) different ways to fill out the bracket. That is

\[ 2^{63} = 9,223,372,036,854,775,808. \]