CHAPTER 13

Moment generating functions

13.1. Definition and examples

Definition (Moment generating function)

The *moment generating function* (MGF) of a random variable X is a function $m_X(t)$ defined by

$$m_X(t) = \mathbb{E}e^{tX},$$

provided the expectation is finite.

In the discrete case m_X is equal to $\sum_x e^{tx}p(x)$ and in the continuous case $\int_{-\infty}^{\infty} e^{tx}f(x)dx$.

Let us compute the moment generating function for some of the distributions we have been working with.

Example 13.1 (Bernoulli)**.**

$$m_X(t) = e^{0\cdot t}(1-p) + e^{1\cdot t}p = e^t p + 1 - p.$$

Example 13.2 (Binomial)**.** Using independence,

$$\mathbb{E}e^{t\sum X_i} = \mathbb{E}\prod e^{tX_i} = \prod \mathbb{E}e^{tX_i} = (pe^t + (1-p))^n,$$

where the X_i are independent Bernoulli random variables. Equivalently

$$\sum_{k=0}^{n} e^{tk}\binom{n}{k}p^k(1-p)^{n-k} = \sum_{k=0}^{n}\binom{n}{k}\left(pe^t\right)^k(1-p)^{n-k} = \left(pe^t + (1-p)\right)^n$$

by the Binomial formula.

Example 13.3 (Poisson)**.**

$$\mathbb{E}e^{tX} = \sum_{k=0}^{\infty}\frac{e^{tk}e^{-\lambda}\lambda^k}{k!} = e^{-\lambda}\sum_{k=0}^{\infty}\frac{(\lambda e^t)^k}{k!} = e^{-\lambda}e^{\lambda e^t} = e^{\lambda(e^t-1)}.$$

Example 13.4 (Exponential)**.**

$$\mathbb{E}e^{tX} = \int_0^{\infty} e^{tx}\lambda e^{-\lambda x}dx = \frac{\lambda}{\lambda - t}$$

if $t < \lambda$, and ∞ if $t \geqslant \lambda$.

Example 13.5 (Standard normal). Suppose $Z \sim \mathcal{N}(0,1)$, then

$$m_Z(t) = \frac{1}{\sqrt{2\pi}} \int e^{tx} e^{-x^2/2} dx = e^{t^2/2} \frac{1}{\sqrt{2\pi}} \int e^{-(x-t)^2/2} dx = e^{t^2/2}.$$

Example 13.6 (General normal). Suppose $X \sim \mathcal{N}(\mu, \sigma^2)$, then we can write $X = \mu + \sigma Z$, and therefore

$$m_X(t) = \mathbb{E}e^{tX} = \mathbb{E}e^{t\mu} e^{t\sigma Z} = e^{t\mu} m_Z(t\sigma) = e^{t\mu} e^{(t\sigma)^2/2} = e^{t\mu + t^2\sigma^2/2}.$$

Proposition 13.1

Suppose $X_1, ..., X_n$ are n independent random variables, and the random variable Y is defined by

$$Y = X_1 + ... + X_n.$$

Then

$$m_Y(t) = m_{X_1}(t) \cdot ... \cdot m_{X_n}(t).$$

Proof. By independence of X and Y and Proposition 12.1 we have

$$m_Y(t) = \mathbb{E}\left(e^{tX_1} \cdot ... \cdot e^{tX_n}\right) = \mathbb{E}e^{tX_1} \cdot ... \cdot \mathbb{E}e^{tX_n} = m_{X_1}(t) \cdot ... \cdot m_{X_n}(t).$$

□

Proposition 13.2

Suppose for two random variables X and Y we have $m_X(t) = m_Y(t) < \infty$ for all t in an interval, then X and Y have the same distribution.

We will not prove this, but this statement is essentially the uniqueness of the Laplace transform $\mathcal{L}$. Recall that the *Laplace transform* of a function $f(x)$ defined for all positive real numbers $s \geqslant 0$

$$(\mathcal{L}f)(s) := \int_0^\infty f(x) e^{-sx} dx$$

Thus if X is a continuous random variable with the PDF such that $f_X(x) = 0$ for $x < 0$, then

$$\int_0^\infty e^{tx} f_X(x) dx = \mathcal{L}f_X(-t).$$

Proposition 13.1 allows to show some of the properties of sums of independent random variables we proved or stated before.

Example 13.7 (Sums of independent normal random variables). If $X \sim \mathcal{N}(a, b^2)$ and $Y \sim \mathcal{N}(c, d^2)$, and X and Y are independent, then by Proposition 13.1

$$m_{X+Y}(t) = e^{at + b^2t^2/2} e^{ct + d^2t^2/2} = e^{(a+c)t + (b^2+d^2)t^2/2},$$

which is the moment generating function for $\mathcal{N}(a+c, b^2+d^2)$. Therefore Proposition 13.2 implies that $X+Y \sim \mathcal{N}(a+c, b^2+d^2)$.

Example 13.8 (Sums of independent Poisson random variables)**.** Similarly, if X and Y are independent Poisson random variables with parameters a and b, respectively, then

$$m_{X+Y}(t) = m_X(t)m_Y(t) = e^{a(e^t-1)}e^{b(e^t-1)} = e^{(a+b)(e^t-1)},$$

which is the moment generating function of a Poisson with parameter $a+b$, therefore $X+Y$ is a Poisson random variable with parameter $a+b$.

One problem with the moment generating function is that it might be infinite. One way to get around this, at the cost of considerable work, is to use the *characteristic function* $\varphi_X(t) = \mathbb{E}e^{itX}$, where $i = \sqrt{-1}$. This is always finite, and is the analogue of the *Fourier transform*.

Definition

Joint MGF The *joint moment generating function* of X and Y is

$$m_{X,Y}(s,t) = \mathbb{E}e^{sX+tY}.$$

If X and Y are independent, then

$$m_{X,Y}(s,t) = m_X(s)m_Y(t)$$

by Proposition 13.2. We will not prove this, but the converse is also true: if $m_{X,Y}(s,t) = m_X(s)m_Y(t)$ for *all* s and t, then X and Y are independent.

13.2. Further examples and applications

Example 13.9. Suppose that m.g.f of X is given by $m(t) = e^{3(e^t-1)}$. Find $\mathbb{P}(X = 0)$.

Solution: We can match this MGF to a known MGF of one of the distributions we considered and then apply Proposition 13.2. Observe that $m(t) = e^{3(e^t-1)} = e^{\lambda(e^t-1)}$, where $\lambda = 3$. Thus $X \sim Poisson(3)$, and therefore

$$\mathbb{P}(X = 0) = e^{-\lambda}\frac{\lambda^0}{0!} = e^{-3}.$$

This example is an illustration to why $m_X(t)$ is called the *moment generating function*. Namely we can use it to find all the moments of X by differentiating $m(t)$ and then evaluating at $t = 0$. Note that

$$m'(t) = \frac{d}{dt}\mathbb{E}\left[e^{tX}\right] = \mathbb{E}\left[\frac{d}{dt}e^{tX}\right] = \mathbb{E}\left[Xe^{tX}\right].$$

Now evaluate at $t = 0$ to get

$$m'(0) = \mathbb{E}\left[Xe^{0\cdot X}\right] = \mathbb{E}[X].$$

Similarly

$$m''(t) = \frac{d}{dt}\mathbb{E}\left[Xe^{tX}\right] = \mathbb{E}\left[X^2e^{tX}\right],$$

so that

$$m''(0) = \mathbb{E}\left[X^2e^0\right] = \mathbb{E}\left[X^2\right].$$

Continuing to differentiate the MGF we have the following proposition.

Proposition 13.3 (Moments from MGF)

For all $n \geqslant 0$ we have

$$\mathbb{E}[X^n] = m^{(n)}(0).$$

Example 13.10. Suppose X is a discrete random variable and has the MGF

$$m_X(t) = \frac{1}{7}e^{2t} + \frac{3}{7}e^{3t} + \frac{2}{7}e^{5t} + \frac{1}{7}e^{8t}.$$

What is the PMF of X? Find $\mathbb{E}X$.

Solution: this does not match any of the known MGFs directly. Reading off from the MGF we guess

$$\frac{1}{7}e^{2t} + \frac{3}{7}e^{3t} + \frac{2}{7}e^{5t} + \frac{1}{7}e^{8t} = \sum_{i=1}^{4} e^{tx_i}p(x_i)$$

then we can take $p(2) = \frac{1}{7}, p(3) = \frac{3}{7}, p(5) = \frac{2}{7}$ and $p(8) = \frac{1}{7}$. Note that these add up to 1, so this is indeed a PMF.

To find $\mathbb{E}[X]$ we can use Proposition 13.3 by taking the derivative of the moment generating function as follows.

$$m'(t) = \frac{2}{7}e^{2t} + \frac{9}{7}e^{3t} + \frac{10}{7}e^{5t} + \frac{8}{7}e^{8t},$$

so that

$$\mathbb{E}[X] = m'(0) = \frac{2}{7} + \frac{9}{7} + \frac{10}{7} + \frac{8}{7} = \frac{29}{7}.$$

Example 13.11. Suppose X has the MGF

$$m_X(t) = (1-2t)^{-\frac{1}{2}} \text{ for } t < \frac{1}{2}.$$

Find the first and second moments of X.

Solution: We have

$$m'_X(t) = -\frac{1}{2}(1-2t)^{-\frac{3}{2}}(-2) = (1-2t)^{-\frac{3}{2}},$$
$$m''_X(t) = -\frac{3}{2}(1-2t)^{-\frac{5}{2}}(-2) = 3(1-2t)^{-\frac{5}{2}}.$$

So that

$$\mathbb{E}X = m'_X(0) = (1-2\cdot 0)^{-\frac{3}{2}} = 1,$$
$$\mathbb{E}X^2 = m''_X(0) = 3(1-2\cdot 0)^{-\frac{5}{2}} = 3.$$

13.3. Exercises

Exercise 13.1. Suppose that you have a fair 4-sided die, and let X be the random variable representing the value of the number rolled.

(a) Write down the moment generating function for X.
(b) Use this moment generating function to compute the first and second moments of X.

Exercise 13.2. Let X be a random variable whose probability density function is given by

$$f_X(x) = \begin{cases} e^{-2x} + \frac{1}{2}e^{-x} & x > 0 \\ 0 & \text{otherwise} \end{cases}.$$

(a) Write down the moment generating function for X.
(b) Use this moment generating function to compute the first and second moments of X.

Exercise 13.3. Suppose that a mathematician determines that the revenue the UConn Dairy Bar makes in a week is a random variable, X, with moment generating function

$$m_X(t) = \frac{1}{(1-2500t)^4}$$

Find the standard deviation of the revenue the UConn Dairy bar makes in a week.

Exercise 13.4. Let X and Y be two independent random variables with respective moment generating functions

$$m_X(t) = \frac{1}{1-5t}, \quad \text{if } t < \frac{1}{5}, \quad m_Y(t) = \frac{1}{(1-5t)^2}, \quad \text{if } t < \frac{1}{5}.$$

Find $\mathbb{E}(X+Y)^2$.

Exercise 13.5. Suppose X and Y are independent Poisson random variables with parameters λ_x, λ_y, respectively. Find the distribution of $X+Y$.

Exercise 13.6. True or False? If $X \sim \operatorname{Exp}(\lambda_x)$ and $Y \sim \operatorname{Exp}(\lambda_y)$ then $X+Y \sim \operatorname{Exp}(\lambda_x + \lambda_y)$. Justify your answer.

13.4. Selected solutions

Solution to Exercise 13.1(A):

$$\begin{aligned} m_X(t) = \mathbb{E}\left[e^{tX}\right] &= e^{1\cdot t}\frac{1}{4} + e^{2\cdot t}\frac{1}{4} + e^{3\cdot t}\frac{1}{4} + e^{4\cdot t}\frac{1}{4} \\ &= \frac{1}{4}\left(e^{1\cdot t} + e^{2\cdot t} + e^{3\cdot t} + e^{4\cdot t}\right) \end{aligned}$$

Solution to Exercise 13.1(B): We have

$$\begin{aligned} m_X'(t) &= \frac{1}{4}\left(e^{1\cdot t} + 2e^{2\cdot t} + 3e^{3\cdot t} + 4e^{4\cdot t}\right), \\ m_X''(t) &= \frac{1}{4}\left(e^{1\cdot t} + 4e^{2\cdot t} + 9e^{3\cdot t} + 16e^{4\cdot t}\right), \end{aligned}$$

so

$$\mathbb{E}X = m_X'(0) = \frac{1}{4}(1+2+3+4) = \frac{5}{2}$$

and

$$\mathbb{E}X^2 = m_X''(0) = \frac{1}{4}(1+4+9+16) = \frac{15}{2}.$$

Solution to Exercise 13.2(A): for $t < 1$ we have

$$\begin{aligned} m_X(t) = \mathbb{E}\left[e^{tX}\right] &= \int_0^\infty e^{tx}\left(e^{-2x} + \frac{1}{2}e^{-x}\right)dx \\ &= \left.\frac{1}{t-2}e^{tx-2x} + \frac{1}{2(t-1)}e^{tx-x}\right|_{x=0}^{x=\infty} = \\ &= 0 - \frac{1}{2-t} + 0 - \frac{1}{2(t-1)} \\ &= \frac{1}{t-2} + \frac{1}{2(1-t)} = \frac{t}{2(2-t)(1-t)} \end{aligned}$$

Solution to Exercise 13.2(B): We have

$$\begin{aligned} m_X'(t) &= \frac{1}{(2-t)^2} + \frac{1}{2(1-t)^2} \\ m_X''(t) &= \frac{2}{(2-t)^3} + \frac{1}{(1-t)^3} \end{aligned}$$

and so $\mathbb{E}X = m_X'(0) = \frac{3}{4}$ and $\mathbb{E}X^2 = m_X'' = \frac{5}{4}$.

Solution to Exercise 13.3: We have $\mathrm{SD}(X) = \sqrt{\mathrm{Var}(X)}$ and $\mathrm{Var}(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2$. Therefore we can compute

$$\begin{aligned}
m'(t) &= 4(2500)(1-2500t)^{-5}, \\
m''(t) &= 20(2500)^2(1-2500t)^{-6}, \\
\mathbb{E}X &= m'(0) = 10,000 \\
\mathbb{E}X^2 &= m''(0) = 125,000,000 \\
\mathrm{Var}(X) &= 125,000,000 - 10,000^2 = 25,000,000 \\
\mathrm{SD}(X) &= \sqrt{25,000,000} = \mathbf{5,000}.
\end{aligned}$$

Solution to Exercise 13.4: First recall that if we let $W = X + Y$, and using that X, Y are independent, then we see that

$$m_W(t) = m_{X+Y}(t) = m_X(t)m_Y(t) = \frac{1}{(1-5t)^3},$$

recall that $\mathbb{E}[W^2] = m_W''(0)$, which we can find from

$$\begin{aligned}
m_W'(t) &= \frac{15}{(1-5t)^4}, \\
m_W''(t) &= \frac{300}{(1-5t)^5},
\end{aligned}$$

thus

$$\mathbb{E}\left[W^2\right] = m_W''(0) = \frac{300}{(1-0)^5} = 300.$$

Solution to Exercise 13.5: Since $X \sim \mathrm{Pois}(\lambda_x)$ and $Y \sim \mathrm{Pois}(\lambda_y)$ then

$$\begin{aligned}
m_X(t) &= e^{\lambda_x\left(e^t-1\right)}, \\
m_Y(t) &= e^{\lambda_y\left(e^t-1\right)}.
\end{aligned}$$

Then

$$\begin{aligned}
m_{X+Y}(t) &= m_X(t)m_Y(t) \\
&\overset{independence}{=} e^{\lambda_x\left(e^t-1\right)}e^{\lambda_y\left(e^t-1\right)} = e^{(\lambda_x+\lambda_y)\left(e^t-1\right)}.
\end{aligned}$$

Thus $X + Y \sim \mathrm{Pois}(\lambda_x + \lambda_y)$.

Solution to Exercise 13.6: We will use Proposition 13.2. Namely, we first find the MGF of $X + Y$ and compare it to the MGF of a random variable $V \sim \mathrm{Exp}(\lambda_x + \lambda_y)$. The MGF of V is

$$m_V(t) = \frac{\lambda_x + \lambda_y}{\lambda_x + \lambda_y - t} \text{ for } t < \lambda_x + \lambda_y.$$

By independence of X and Y

$$m_{X+Y}(t) = m_X(t)m_Y(t) = \frac{\lambda_x}{\lambda_x - t} \cdot \frac{\lambda_y}{\lambda_y - t},$$

but

$$\frac{\lambda_x + \lambda_y}{\lambda_x + \lambda_y - t} \neq \frac{\lambda_x}{\lambda_x - t} \cdot \frac{\lambda_y}{\lambda_y - t}$$

and hence the statement is false.