CHAPTER 14

Limit laws

14.1. SLLN, WLLN, Chebyshev's inequality, CLT

Suppose we have a probability space of a sample space S, σ-field $\mathcal{F}$ and probability $\mathbb{P}$ defined on $\mathcal{F}$. We consider a collection of random variables X_i defined on the same probability space.

Definition (Sums of i.i.d. random variables)

We say the sequence of random variables $\{X_i\}_{i=1}^{\infty}$ are *i.i.d.* (independent and identically distributed) if they are (mutually) independent and all have the same distribution. We call $S_n = \sum_{i=1}^{n} X_i$ the *partial sum process*.

In the case of continuous or discrete random variables, having the same distribution means that they all have the same probability density.

Theorem 14.1 (Strong law of large numbers (SLLN))

Suppose $\{X_i\}_{i=1}^{\infty}$ is a sequence of i.i.d. random variables with $\mathbb{E}|X_i| < \infty$, and let $\mu = \mathbb{E}X_i$. Then

$$\frac{S_n}{n} \xrightarrow[n\to\infty]{} \mu.$$

The convergence here means that $S_n(\omega)/n \to \mu$ for every $\omega \in S$, where S is the probability space, except possibly for a set of ω of probability 0.

The proof of Theorem 14.1 is quite hard, and we prove a weaker version, the weak law of large numbers (WLLN).

Theorem (Weak law of large numbers (WLLN))

Suppose $\{X_i\}_{i=1}^{\infty}$ is a sequence of i.i.d. random variables with finite first and second moments, that is, $\mathbb{E}|X_1|$ and $\operatorname{Var} X_1$ are finite.
For every $a > 0$,

$$\mathbb{P}\left(\left|\frac{S_n}{n} - \mathbb{E}X_1\right| > a\right) \xrightarrow[n\to\infty]{} 0.$$

It is not even that easy to give an example of random variables that satisfy the WLLN but not the SLLN. Before proving the WLLN, we need an inequality called *Chebyshev's inequality*.

Proposition 14.1 (Chebyshev's inequality)

If a random variable $Y \geqslant 0$, then for any A

$$\mathbb{P}(Y > A) \leqslant \frac{\mathbb{E}Y}{A}.$$

Proof. We only consider the case for continuous densities, the case for discrete densities being similar. We have

$$\begin{aligned}\mathbb{P}(Y > A) &= \int_A^\infty f_Y(y)\,dy \leqslant \int_A^\infty \frac{y}{A} f_Y(y)\,dy \\ &\leqslant \frac{1}{A}\int_{-\infty}^\infty y f_Y(y)\,dy = \frac{1}{A}\mathbb{E}Y.\end{aligned}$$

$\square$

We now prove the WLLN.

Proof of Theorem 14.1. Recall $\mathbb{E}S_n = n\mathbb{E}X_1$, and by the independence $\operatorname{Var} S_n = n \operatorname{Var} X_1$, so $\operatorname{Var}(S_n/n) = \operatorname{Var} X_1/n$. We have

$$\begin{aligned}\mathbb{P}\left(\left|\frac{S_n}{n} - \mathbb{E}X_1\right| > a\right) &= \mathbb{P}\left(\left|\frac{S_n}{n} - \mathbb{E}\left(\frac{S_n}{n}\right)\right| > a\right) \\ &= \mathbb{P}\left(\left|\frac{S_n}{n} - \mathbb{E}\left(\frac{S_n}{n}\right)\right|^2 > a^2\right) \overset{\textit{Chebyshev}}{\leqslant} \frac{\mathbb{E}\left|\frac{S_n}{n} - \mathbb{E}(\frac{S_n}{n})\right|^2}{a^2} \\ &= \frac{\operatorname{Var}(\frac{S_n}{n})}{a^2} = \frac{\frac{\operatorname{Var} X_1}{n}}{a^2} \to 0.\end{aligned}$$

The inequality step follows from Chebyshev's inequality (Proposition 14.1) with $A = a^2$ and $Y = \left|\frac{S_n}{n} - \mathbb{E}(\frac{S_n}{n})\right|^2$. $\square$

We now turn to the central limit theorem (CLT).

Theorem 14.2 (CLT)

Suppose $\{X_i\}_{i=1}^\infty$ is a sequence of i.i.d. random variables such that $\mathbb{E}X_i^2 < \infty$. Let $\mu = \mathbb{E}X_i$ and $\sigma^2 = \operatorname{Var} X_i$. Then

$$\mathbb{P}\left(a \leqslant \frac{S_n - n\mu}{\sigma\sqrt{n}} \leqslant b\right) \xrightarrow[n\to\infty]{} \mathbb{P}(a \leqslant Z \leqslant b)$$

for every a and b, where $Z \sim \mathcal{N}(0,1)$.

The ratio on the left is $(S_n - \mathbb{E}S_n)/\sqrt{\operatorname{Var} S_n}$. We do not claim that this ratio converges for any $\omega \in S$ (in fact, it does not), but that the probabilities converge.

Example 14.1 (Theorem 9.1). If X_i are i.i.d. Bernoulli random variables, so that S_n is a binomial, this is just the normal approximation to the binomial as in Theorem 9.1.

Example 14.2. Suppose we roll a die 3600 times. Let X_i be the number showing on the i^{th} roll. We know S_n/n will be close to 3.5. What is the (approximate) probability it differs from 3.5 by more than 0.05?

Solution: We want to estimate

$$\mathbb{P}\left(\left|\frac{S_n}{n} - 3.5\right| > 0.05\right).$$

We rewrite this as

$$\mathbb{P}(|S_n - n\mathbb{E}X_1| > (0.05)(3600)) = \mathbb{P}\left(\left|\frac{S_n - n\mathbb{E}X_1}{\sqrt{n}\sqrt{\operatorname{Var} X_1}}\right| > \frac{180}{(60)\sqrt{\frac{35}{12}}}\right)$$

Note that $\frac{180}{(60)\sqrt{\frac{35}{12}}} \approx 1.7566$, so this probability can be approximately as

$$\mathbb{P}(|Z| > 1.757) \approx 0.08.$$

Example 14.3. Suppose the lifetime of a human has expectation 72 and variance 36. What is the (approximate) probability that the average of the lifetimes of 100 people exceeds 73?

Solution: We want to estimate

$$\begin{aligned}\mathbb{P}\left(\frac{S_n}{n} > 73\right) &= \mathbb{P}(S_n > 7300) \\ &= \mathbb{P}\left(\frac{S_n - n\mathbb{E}X_1}{\sqrt{n}\sqrt{\operatorname{Var} X_1}} > \frac{7300 - (100)(72)}{\sqrt{100}\sqrt{36}}\right) \\ &\approx \mathbb{P}(Z > 1.667) \approx 0.047.\end{aligned}$$

Sketch of the proof of Theorem 14.2. The idea behind proving the central limit theorem is the following. It turns out that if

$$m_{Y_n}(t) \to m_Z(t) \text{ for every } t,$$

then $\mathbb{P}(a \leqslant Y_n \leqslant b) \to \mathbb{P}(a \leqslant Z \leqslant b)$, though *we won't prove this.*

We denote

$$\begin{aligned}Y_n &= \frac{S_n - n\mu}{\sigma\sqrt{n}}, \\ W_i &= \frac{X_i - \mu}{\sigma}.\end{aligned}$$

Then $\mathbb{E}W_i = 0$, $\operatorname{Var} W_i = \frac{\operatorname{Var} X_i}{\sigma^2} = 1$, the random variables W_i are (mutually) independent, and

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} = \frac{\sum_{i=1}^n W_i}{\sqrt{n}}.$$

So there is no loss of generality in assuming that $\mu = 0$ and $\sigma = 1$. Then

$$m_{Y_n}(t) = \mathbb{E}e^{tY_n} = \mathbb{E}e^{(t/\sqrt{n})(S_n)} = m_{S_n}(t/\sqrt{n}).$$

Since the random variables X_i are i.i.d., all the X_i have the same moment generating function. Thus for $S_n = X_1 + ... + X_n$, we have

$$m_{S_n}(t) = m_{X_1}(t) \cdots m_{X_n}(t) = [m_{X_1}(t)]^n.$$

If we expand e^{tX_1} as a power series in t, we have

$$m_{X_1}(t) = \mathbb{E}e^{tX_1} = 1 + t\mathbb{E}X_1 + \frac{t^2}{2!}\mathbb{E}(X_1)^2 + \frac{t^3}{3!}\mathbb{E}(X_1)^3 + \cdots .$$

We put the above together and obtain

$$\begin{aligned} m_{Y_n}(t) &= m_{S_n}\left(\frac{t}{\sqrt{n}}\right) = [m_{X_1}(t/\sqrt{n})]^n \\ &= \left(1 + t \cdot 0 + \frac{(t/\sqrt{n})^2}{2!} + R_n\right)^n = \left(1 + \frac{t^2}{2n} + R_n\right)^n, \end{aligned}$$

where $|R_n|/n \xrightarrow[n\to\infty]{} 0$. Using the fact that

$$1 + \frac{a}{n} \xrightarrow[n\to\infty]{} e^a,$$

we see that $\left(1 + \frac{t^2}{2n} + R_n\right)^n \xrightarrow[n\to\infty]{} e^{t^2/2} = m_Z(t)$, and so we can use Proposition 13.2 to conclude the proof. $\square$

14.2. Further examples and applications

Example 14.4. If 10 fair dice are rolled, find the approximate probability that the sum obtained is between 30 and 40, inclusive.

Solution: We will need to use the ± 0.5 continuity correction because these are discrete random variables. Let X_i denote the value of the ith die. Recall that

$$\mathbb{E}\left(X_i\right) = \frac{7}{2} \operatorname{Var}(X_i) = \frac{35}{12}.$$

Take

$$X = X_1 + \cdots + X_n$$

to be their sum. To apply the CLT we have

$$\begin{aligned} n\mu &= 10 \cdot \frac{7}{2} = 35 \\ \sigma\sqrt{n} &= \sqrt{\frac{350}{12}}, \end{aligned}$$

thus using the continuity correction we have

$$\begin{aligned} \mathbb{P}\left(29.5 \leqslant X \leqslant 40.5\right) &= \mathbb{P}\left(\frac{29.5 - 35}{\sqrt{\frac{350}{12}}} \leqslant \frac{X - 35}{\sqrt{\frac{350}{12}}} \leqslant \frac{40.5 - 35}{\sqrt{\frac{350}{12}}}\right) \\ &\approx \mathbb{P}\left(-1.0184 \leqslant Z \leqslant 1.0184\right) \\ &= \Phi\left(1.0184\right) - \Phi\left(-1.0184\right) \\ &= 2\Phi\left(1.0184\right) - 1 = 0.692. \end{aligned}$$

Example 14.5. Your instructor has 1000 Probability final exams that needs to be graded. The time required to grade an exam are all i.i.d. with mean of 20 minutes and standard deviation of 4 minutes. Approximate the probability that your instructor will be able to grade at least 25 exams in the first 450 minutes of work.

Solution: Let X_i be the time it takes to grade exam i. Then

$$X = X_1 + \cdots + X_{25}$$

is the time it takes to grade the first 25 exams. We want $\mathbb{P}\left(X \leq 450\right)$. To apply the CLT we have

$$\begin{aligned} n\mu &= 25 \cdot 20 = 500 \\ \sigma\sqrt{n} &= 4\sqrt{25} = 20. \end{aligned}$$

Thus

$$\begin{aligned} \mathbb{P}\left(X \leqslant 450\right) &= \mathbb{P}\left(\frac{X - 500}{20} \leqslant \frac{450 - 500}{20}\right) \\ &\approx \mathbb{P}\left(Z \leqslant -2.5\right) \\ &= 1 - \Phi(2.5) = 0.006. \end{aligned}$$

14.3. Exercises

Exercise 14.1. In a 162-game season, find the approximate probability that a team with a 0.5 chance of winning will win at least 87 games.

Exercise 14.2. An individual students MATH 3160 Final exam score at UConn is a random variable with mean 75 and variance 25, How many students would have to take the examination to ensure with probability at least 0.9 that the class average would be within 5 of 75?

Exercise 14.3. Let $X_1, X_2, ..., X_{100}$ be independent exponential random variables with parameter $\lambda = 1$. Use the central limit theorem to approximate

$$\mathbb{P}\left(\sum_{i=1}^{100} X_i > 90\right).$$

Exercise 14.4. Suppose an insurance company has 10,000 automobile policy holders. The expected yearly claim per policy holder is \$240, with a standard deviation of \$800. Approximate the probability that the total yearly claim is greater than \$2,500,000.

Exercise 14.5. Suppose that the checkout time at the UConn dairy bar has a mean of 5 minutes and a standard deviation of 2 minutes. Estimate the probability to serve at least 36 customers during a 3-hour and a half shift.

Exercise 14.6. Shabazz Napier is a basketball player in the NBA. His expected number of points per game is 15 with a standard deviation of 5 points per game. The NBA season is 82 games long. Shabazz is guaranteed a ten million dollar raise next year if he can score a total of 1300 points this season. Approximate the probability that Shabazz will get a raise next season.

14.4. Selected solutions

Solution to Exercise 14.1: Let X_i be 1 if the team wins the ith game and 0 if the team loses. This is a Bernoulli random variable with $p = 0.5$. Thus $\mu = p = 0.5$ and $\sigma^2 = p(1-p) = (0.5)^2$. Then

$$X = \sum_{i=1}^{162} X_i$$

is the number of games won in the season. Using the CLT with

$$n\mu = 162 \cdot 0.5 = 81$$
$$\sigma\sqrt{n} = 0.5\sqrt{162} \approx 6.36,$$

then

$$\begin{aligned}\mathbb{P}\left(\sum_{i=1}^{162} X_i \geqslant 87\right) &= \mathbb{P}(X \geqslant 86.5) \\ &= \mathbb{P}\left(\frac{X-81}{6.36} > \frac{86.5-81}{6.36}\right) \\ &\approx \mathbb{P}(Z > 0.86) \approx 0.1949,\end{aligned}$$

where we used a continuity correction since X is a discrete random variable.

Solution to Exercise 14.2: Now $\mu = 75$, $\sigma^2 = 25, \sigma = 5$.

$$\begin{aligned}\mathbb{P}\left(70 < \frac{\sum_{i=1}^n X_i}{n} < 80\right) \geqslant 0.9 &\Longleftrightarrow \mathbb{P}\left(70 \cdot n < \sum_{i=1}^n X_i < 80 \cdot n\right) \geqslant 0.9 \\ &\Longleftrightarrow \mathbb{P}\left(\frac{70 \cdot n - 75 \cdot n}{5\sqrt{n}} < Z < \frac{80 \cdot n - 75 \cdot n}{5\sqrt{n}}\right) \geqslant 0.9 \\ &\Longleftrightarrow \mathbb{P}\left(-5\frac{\sqrt{n}}{5} < Z < 5\frac{\sqrt{n}}{5}\right) \geqslant 0.9 \\ &\Longleftrightarrow \mathbb{P}\left(-\sqrt{n} < Z < \sqrt{n}\right) \geqslant 0.9 \\ &\Longleftrightarrow \Phi\left(\sqrt{n}\right) - \Phi\left(-\sqrt{n}\right) \geqslant 0.9 \\ &\Longleftrightarrow \Phi\left(\sqrt{n}\right) - \left(1 - \Phi\left(\sqrt{n}\right)\right) \geqslant 0.9 \\ &\Longleftrightarrow 2\Phi\left(\sqrt{n}\right) - 1 \geqslant 0.9 \\ &\Longleftrightarrow \Phi\left(\sqrt{n}\right) \geqslant 0.95.\end{aligned}$$

Using the table inversely we have that

$$\sqrt{n} \geqslant 1.65 \Longrightarrow n \geqslant 2.722$$

hence the first integer that insurers that $n \geqslant 2.722$ is

$$n = 3.$$

Solution to Exercise 14.3: Since $\lambda = 1$ then $\mathbb{E}X_i = 1$ and $\operatorname{Var}(X_i) = 1$. Use the CLT with

$$\begin{aligned} n\mu &= 100 \cdot 1 = 100 \\ \sigma\sqrt{n} &= 1 \cdot \sqrt{100} = 10. \end{aligned}$$

$$\begin{aligned} \mathbb{P}\left(\sum_{i=1}^{100} X_i > 90\right) &= \mathbb{P}\left(\frac{\sum_{i=1}^{100} X_i - 100 \cdot 1}{1 \cdot \sqrt{100}} > \frac{90 - 100 \cdot 1}{1 \cdot \sqrt{100}}\right) \\ &\approx \mathbb{P}(Z > -1) \approx 0.8413. \end{aligned}$$

Solution to Exercise 14.4:

$$\begin{aligned} \mathbb{P}(X \geqslant 1300) &= \mathbb{P}\left(\frac{X - 2400000}{80000} \geqslant \frac{2500000 - 2400000}{80000}\right) \\ &\approx \mathbb{P}(Z \geqslant 1.25) \\ &= 1 - \Phi(1.25) \approx 1 - 0.8944 = 0.1056. \end{aligned}$$

Solution to Exercise 14.5: Let X_i be the time it takes to check out customer i. Then

$$X = X_1 + \cdots + X_{36}$$

is the time it takes to check out 36 customer. We want to estimate $\mathbb{P}(X \leqslant 210)$. Use the CLT with

$$\begin{aligned} n\mu &= 36 \cdot 5 = 180, \\ \sigma\sqrt{n} &= 2\sqrt{36} = 12. \end{aligned}$$

Thus

$$\begin{aligned} \mathbb{P}(X \leqslant 210) &= \mathbb{P}\left(\frac{X - 180}{12} \leqslant \frac{210 - 180}{12}\right) \\ &\approx \mathbb{P}(Z \leqslant 2.5) \\ &= \Phi(2.5) \approx 0.9938. \end{aligned}$$

Solution to Exercise 14.6:

Let X_i be the number of points scored by Shabazz in game i . Then

$$X = X_1 + \cdots + X_{82}$$

is the total number of points in a whole season. We want to estimate $\mathbb{P}(X \geqslant 1800)$. Use the CLT with

$$\begin{aligned} n\mu &= 82 \cdot 15 = 1230, \\ \sigma\sqrt{n} &= 5\sqrt{82} \approx 45.28. \end{aligned}$$

Thus

$$\begin{aligned}\mathbb{P}(X \geqslant 1300) &= \mathbb{P}\left(\frac{X-1230}{45.28} \geqslant \frac{1300-1230}{45.28}\right)\\ &\approx \mathbb{P}(Z \geqslant 1.55)\\ &= 1-\Phi(1.55) \approx 1-0.9394 = 0.0606.\end{aligned}$$