CHAPTER 14

Limit laws

14.1. Introduction

Suppose $X_i$ are independent and have the same distribution. In the case of continuous or discrete random variables, this means they all have the same density. We say the $X_i$ are i.i.d., which stands for “independent and identically distributed.” Let $S_n = \sum_{i=1}^{n} X_i$. $S_n$ is called the partial sum process.

**Theorem 14.1:** Suppose $\mathbb{E}|X_i| < \infty$ and let $\mu = \mathbb{E}X_i$. Then

$$\frac{S_n}{n} \rightarrow \mu.$$ 

This is known as the strong law of large numbers (SLLN). The convergence here means that $S_n(\omega)/n \rightarrow \mu$ for every $\omega \in S$, where $S$ is the probability space, except possibly for a set of $\omega$ of probability 0.

The proof of Theorem 14.1 is quite hard, and we prove a weaker version, the weak law of large numbers (WLLN). The WLLN states that for every $a > 0$,

$$\mathbb{P}\left(\left|\frac{S_n}{n} - \mathbb{E}X_1\right| > a\right) \rightarrow 0$$

as $n \rightarrow \infty$. It is not even that easy to give an example of random variables that satisfy the WLLN but not the SLLN.

Before proving the WLLN, we need an inequality called Chebyshev’s inequality.

**Proposition 14.2:** If $Y \geq 0$, then for any $A$,

$$\mathbb{P}(Y > A) \leq \frac{\mathbb{E}Y}{A}.$$ 

**Proof.** We do the case for continuous densities, the case for discrete densities being similar. We have

$$\mathbb{P}(Y > A) = \int_{A}^{\infty} f_Y(y) \, dy \leq \int_{A}^{\infty} \frac{y}{A} f_Y(y) \, dy$$

$$\leq \frac{1}{A} \int_{-\infty}^{\infty} y f_Y(y) \, dy = \frac{1}{A} \mathbb{E}Y.$$ 

$\square$
We now prove the WLLN.

**Theorem 14.3:** Suppose the $X_i$ are i.i.d. and $E|X_1|$ and $VarX_1$ are finite. Then for every $a > 0$,
\[
P\left(\left|\frac{S_n}{n} - E X_1\right| > a\right) \rightarrow 0
\]
as $n \rightarrow \infty$.

**Proof.** Recall $E S_n = nE X_1$ and by the independence, $VarS_n = nVarX_1$, so $Var(S_n/n) = VarX_1/n$. We have
\[
P\left(\left|\frac{S_n}{n} - E X_1\right| > a\right) = \mathbb{P}\left(\left|\frac{S_n}{n} - E \left(\frac{S_n}{n}\right)\right| > a\right)
\]
\[
= \mathbb{P}\left(\left|\frac{S_n}{n} - E \left(\frac{S_n}{n}\right)^2\right| > a^2\right)
\]
\[
\leq \frac{Var\left(\frac{S_n}{n}\right)}{a^2}
\]
\[
= \frac{VarX_1}{a^2} \rightarrow 0.
\]
The inequality step follows from Proposition 14.2 with $A = a^2$ and $Y = |\frac{S_n}{n} - E \left(\frac{S_n}{n}\right)|^2$. □

We now turn to the central limit theorem (CLT).

**Theorem 14.4:** Suppose the $X_i$ are i.i.d. Suppose $E X_i^2 < \infty$. Let $\mu = E X_i$ and $\sigma^2 = VarX_i$. Then
\[
P\left(a \leq \frac{S_n - n\mu}{\sigma\sqrt{n}} \leq b\right) \rightarrow \mathbb{P}(a \leq Z \leq b)
\]
for every $a$ and $b$, where $Z$ is a $\mathcal{N}(0, 1)$.

The ratio on the left is $(S_n - E S_n)/\sqrt{VarS_n}$. We do not claim that this ratio converges for any $\omega$ (in fact, it doesn’t), but that the probabilities converge.

**Example 14.1:** If the $X_i$ are i.i.d. Bernoulli random variables, so that $S_n$ is a binomial, this is just the normal approximation to the binomial.

**Example 14.2:** Suppose we roll a die 3600 times. Let $X_i$ be the number showing on the $i^{th}$ roll. We know $S_n/n$ will be close to 3.5. What’s the probability it differs from 3.5 by more than 0.05?

**Answer.** We want
\[
P\left(\left|\frac{S_n}{n} - 3.5\right| > .05\right).
\]
We rewrite this as
\[ \Pr(|S_n - n\mathbb{E}X_1| > .05(3600)) = \Pr\left(\left| \frac{S_n - n\mathbb{E}X_1}{\sqrt{n}\sqrt{\text{Var}X_1}} \right| > \frac{180}{(60)\sqrt{35/12}} \right) \]
\[ r \approx \Pr(|Z| > 1.756) \approx .08. \]

Example 14.3: Suppose the lifetime of a human has expectation 72 and variance 36. What is the probability that the average of the lifetimes of 100 people exceeds 73?

Answer. We want
\[ \Pr\left(\frac{S_n}{n} > 73\right) = \Pr(S_n > 7300) \]
\[ = \Pr\left(\frac{S_n - n\mathbb{E}X_1}{\sqrt{n}\sqrt{\text{Var}X_1}} > \frac{7300 - (100)(72)}{\sqrt{100\cdot 36}} \right) \]
\[ \approx \Pr(Z > 1.667) \approx .047. \]

The idea behind proving the central limit theorem is the following. It turns out that if \( m_{Y_n}(t) \to m_Z(t) \) for every \( t \), then \( \Pr(a \leq Y_n \leq b) \to \Pr(a \leq Z \leq b) \). (We won’t prove this.) We are going to let \( Y_n = (S_n - n\mu)/\sigma\sqrt{n} \). Let \( W_i = (X_i - \mu)/\sigma \). Then \( \mathbb{E}W_i = 0 \), \( \text{Var}W_i = \frac{\text{Var}X_i}{\sigma^2} = 1 \), the \( W_i \) are independent, and
\[ \frac{S_n - n\mu}{\sigma\sqrt{n}} = \sum_{i=1}^n W_i. \]

So there is no loss of generality in assuming that \( \mu = 0 \) and \( \sigma = 1 \). Then
\[ m_{Y_n}(t) = \mathbb{E}e^{tY_n} = \mathbb{E}e^{(t/\sqrt{n})(S_n)} = m_{S_n}(t/\sqrt{n}). \]

Since the \( X_i \) are i.i.d., all the \( X_i \) have the same moment generating function. Since \( S_n = X_1 + \cdot + X_n \), then
\[ m_{S_n}(t) = m_{X_1}(t) \cdots m_{X_n}(t) = [m_{X_1}(t)]^n. \]

If we expand \( e^{tX_1} \) as a power series,
\[ m_{X_1}(t) = \mathbb{E}e^{tX_1} = 1 + t\mathbb{E}X_1 + \frac{t^2}{2!}\mathbb{E}(X_1)^2 + \frac{t^3}{3!}\mathbb{E}(X_1)^3 + \cdots. \]

We put the above together and obtain
\[ m_{Y_n}(t) = m_{S_n}(t/\sqrt{n}) \]
\[ = [m_{X_1}(t/\sqrt{n})]^n \]
\[ = \left[ 1 + t \cdot 0 + \frac{(t/\sqrt{n})^2}{2!} + R_n \right]^n \]
\[ = \left[ 1 + \frac{t^2}{2n} + R_n \right]^n, \]
where \( |R_n|/n \to 0 \) as \( n \to \infty \). This converges to \( e^{t^2/2} = m_Z(t) \) as \( n \to \infty \).
14.2. Further examples and applications

Example 14.4: If 10 fair dice are rolled, find the approximate probability that the sum obtained is between 30 and 40, inclusive.

Answer. We will need to use the ±.5 continuity correction because these are discrete random variables. Let \( X_i \) denote the value of the \( i \)th die. Recall that
\[
\mathbb{E}(X_i) = \frac{7}{2} \quad \text{Var}(X_i) = \frac{35}{12}.
\]
Take
\[ X = X_1 + \cdots + X_n \]
to be their sum. Using the CLT we need
\[
n\mu = 10 \cdot \frac{7}{2} = 35
\]
\[
\sigma\sqrt{n} = \sqrt{\frac{350}{12}}
\]
thus using the continuity correction, then
\[
\mathbb{P}(29.5 \leq X \leq 40.5) = \mathbb{P}
\left( \frac{29.5 - 35}{\sqrt{\frac{350}{12}}} \leq \frac{X - 35}{\sqrt{\frac{350}{12}}} \leq \frac{40.5 - 35}{\sqrt{\frac{350}{12}}} \right)
\]
\[
\approx \mathbb{P}(-1.0184 \leq Z \leq 1.0184)
\]
\[
= \Phi(1.0184) - \Phi(-1.0184)
\]
\[
= 2\Phi(1.0184) - 1 = .692.
\]

Example 14.5: Your instructor has 1000 Probability final exams that needs to be graded. The time required to grade an exam are all i.i.s. with mean of 20 minutes and standard deviation of 4 minutes. Approximate the probability that your instructor will be able to grade at least 25 exams in the first 450 minutes of work.

Answer. Let \( X_i \) be the time it takes to grade exam \( i \). Then
\[ X = X_1 + \cdots + X_{25} \]
is the time it takes to grade the first 25 exams. We want \( \mathbb{P}(X \leq 450) \). Using the CLT, we need
\[
n\mu = 25 \cdot 20 = 500
\]
\[
\sigma\sqrt{n} = 4\sqrt{25} = 20.
\]
Thus
\[
\mathbb{P}(X \leq 450) = \mathbb{P}
\left( \frac{X - 500}{20} \leq \frac{450 - 500}{20} \right)
\]
\[
\approx \mathbb{P}(Z \leq -2.5)
\]
\[
= 1 - \Phi(2.5)
\]
\[
= .006.
\]
14.3. Exercises

Exercise 14.1: In a 162-game season, find the approximate probability that a team with a 0.5 chance of winning will win at least 87 games.

Exercise 14.2: An individual student's MATH 3160 Final exam score at UConn is a random variable with mean 75 and variance 25. How many students would have to take the examination to ensure with probability at least .9 that the class average would be within 5 of 75?

Exercise 14.3: Let $X_1, X_2, \ldots, X_{100}$ be independent exponential random variables with parameter $\lambda = 1$. Use the central limit theorem to approximate
\[ P \left( \sum_{i=1}^{100} X_i > 90 \right) . \]

Exercise 14.4: Suppose an insurance company has 10,000 automobile policy holders. The expected yearly claim per policy holder is $240, with a standard deviation of $800. Approximate the probability that the total yearly claim is greater than $2,500,000.

Exercise 14.5: Suppose that the checkout time at the UConn dairy bar has a mean of 5 minutes and a standard deviation of 2 minutes. Estimate the probability to serve at least 36 customers during a 3-hour and a half shift.

Exercise 14.6: Shabazz Napier is a basketball player in the NBA. His expected number of points per game is 15 with a standard deviation of 5 points per game. The NBA season is 82 games long. Shabazz is guaranteed a ten million dollar raise next year if he can score a total of 1300 points this season. Approximate the probability that Shabazz will get a raise next season.
14.4. Selected solutions

Solution to Exercise [14.1] Let $X_i$ be 1 if the team wins the $i$th game and 0 if the team loses. This is a Bernoulli R.V. with $p = .5$. Thus $\mu = p = .5$ and $\sigma^2 = p(1 - p) = (.5)^2$. Then

$$X = \sum_{i=1}^{162} X_i$$

is the number of games won in the season. Using CLT

$$n\mu = 162 \cdot .5 = 81$$
$$\sigma\sqrt{n} = .5\sqrt{162} = 6.36$$

then

$$P \left( \sum_{i=1}^{162} X_i \geq 87 \right) = P (X \geq 86.5)$$
$$= P \left( \frac{X - 81}{6.36} > \frac{86.5 - 81}{6.36} \right)$$
$$\approx P (Z > .86) = .1949$$

where we used a correction since $X$ is a discrete r.v.

Solution to Exercise [14.2] Now $\mu = 75$, $\sigma^2 = 25$, $\sigma = 5$.

$$\mathbb{P} \left( 70 < \frac{\sum_{i=1}^{n} X_i}{n} < 80 \right) \geq .9 \iff \mathbb{P} \left( 70 \cdot n < \sum_{i=1}^{n} X_i < 80 \cdot n \right) \geq .9$$
$$\iff \mathbb{P} \left( \frac{70 \cdot n - 75 \cdot n}{5\sqrt{n}} < Z < \frac{80 \cdot n - 75 \cdot n}{5\sqrt{n}} \right) \geq .9$$
$$\iff \mathbb{P} \left( -5 \frac{\sqrt{n}}{5} < Z < 5 \frac{\sqrt{n}}{5} \right) \geq .9$$
$$\iff \mathbb{P} \left( -\sqrt{n} < Z < \sqrt{n} \right) \geq .9$$
$$\iff \Phi (\sqrt{n}) - \Phi (-\sqrt{n}) \geq .9$$
$$\iff \Phi (\sqrt{n}) - (1 - \Phi (\sqrt{n})) \geq .9$$
$$\iff 2\Phi (\sqrt{n}) - 1 \geq .9$$
$$\iff \Phi (\sqrt{n}) \geq .95$$

Using the table inversely we have that

$$\sqrt{n} \geq 1.65 \implies n \geq 2.722$$

hence the first integer that insurers that $n \geq 2.722$ is

$$n = 3.$$ 

Solution to Exercise [14.3] Since $\lambda = 1$ then $\mathbb{E}X_i = 1$ and $\text{Var}(X_i) = 1$. Use CLT

$$n\mu = 100 \cdot 1 = 100$$
$$\sigma\sqrt{n} = 1 \cdot \sqrt{100} = 10.$$
\[
P\left(\sum_{i=1}^{100} X_i > 90\right) = P\left(\frac{\sum_{i=1}^{100} X_i - 100 \cdot 1}{1 \cdot \sqrt{100}} > \frac{90 - 100 \cdot 1}{1 \cdot \sqrt{100}}\right)
\approx P(Z > -1) = .8413.
\]

Solution to Exercise 14.4

\[
P(X \geq 1300) = P\left(\frac{X - 2400000}{80000} \geq \frac{2500000 - 2400000}{80000}\right)
\approx P(Z \geq 1.25)
= 1 - \Phi(1.25) = 1 - .8944
= .1056.
\]

Solution to Exercise 14.5

Let \(X_i\) be the time it takes to check out customer \(i\). Then
\[X = X_1 + \cdots + X_{36}\]
is the time it takes to check out 36 customer. We want \(P(X \leq 210)\). Use CLT,
\[
n\mu = 36 \cdot 5 = 180
\]
\[
\sigma \sqrt{n} = 2\sqrt{36} = 12.
\]
Thus
\[
P(X \leq 210) = P\left(\frac{X - 180}{12} \leq \frac{210 - 180}{12}\right)
\approx P(Z \leq 2.5)
= \Phi(2.5)
= .9938.
\]

Solution to Exercise 14.6

Let \(X_i\) be the number of points scored by Shabazz in game \(i\). Then
\[X = X_1 + \cdots + X_{82}\]
is the total number of points in a whole season. We want \(P(X \geq 1800)\). Use CLT,
\[
n\mu = 82 \cdot 15 = 1230
\]
\[
\sigma \sqrt{n} = 5\sqrt{82} = 45.28.
\]
Thus
\[
P(X \geq 1300) = P\left(\frac{X - 1230}{45.28} \geq \frac{1300 - 1230}{45.28}\right)
\approx P(Z \geq 1.55)
= 1 - \Phi(1.55) = 1 - .9394
= .0606.
\]