CHAPTER 2

The probability set-up

2.1. Basic theory of probability

We will have a *sample space*, denoted by S (sometimes Ω) that consists of all possible outcomes. For example, if we roll two dice, the sample space would be all possible pairs made up of the numbers one through six. An *event* is a subset of S.

Another example is to toss a coin 2 times, and let

$$S = \{HH, HT, TH, TT\};$$

or to let S be the possible orders in which 5 horses finish in a horse race; or S the possible prices of some stock at closing time today; or $S = [0, \infty)$; the age at which someone dies; or S the points in a circle, the possible places a dart can hit. We should also keep in mind that the same setting can be described using different sample set. For example, in two solutions in Example 1.30 we used two different sample sets.

2.1.1. Sets. We start by describing elementary operations on sets. By a *set* we mean a collection of distinct objects called *elements of the set*, and we consider a set as an object in its own right.

Set operations

Suppose S is a set. We say that $A \subset S$, that is, A is a *subset* of S if every element in A is contained in S;
$A \cup B$ is the *union* of sets $A \subset S$ and $B \subset S$ and denotes the points of S that are in A or B or both;
$A \cap B$ is the *intersection* of sets $A \subset S$ and $B \subset S$ and is the set of points that are in both A and B;
$\emptyset$ denotes the *empty set*;
A^c is the *complement* of A, that is, the points in S that are not in A.

We extend this definition to have $\cup_{i=1}^n A_i$ is the union of sets $A_1, \cdots, A_n$, and similarly $\cap_{i=1}^n A_i$.

An exercise is to show that

De Morgan's laws

$$\left(\bigcup_{i=1}^n A_i\right)^c = \bigcap_{i=1}^n A_i^c \quad \text{and} \quad \left(\bigcap_{i=1}^n A_i\right)^c = \bigcup_{i=1}^n A_i^c.$$

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We will also need the notion of *pairwise disjoint sets* $\{E_i\}_{i=1}^{\infty}$ which means that $E_i \cap E_j = \emptyset$ unless $i = j$.

There are no restrictions on the sample space S. The collection of events, $\mathcal{F}$, is assumed to be a σ-field, which means that it satisfies the following.

Definition (σ-field)

A collection $\mathcal{F}$ of sets in S is called a *σ-field* if
(i) Both $\emptyset, S$ are in $\mathcal{F}$,
(ii) if A is in $\mathcal{F}$, then A^c is in $\mathcal{F}$,
(iii) if $A_1, A_2, \ldots$ are in $\mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$ are in $\mathcal{F}$.

Typically we will take $\mathcal{F}$ to be all subsets of S, and so (i)-(iii) are automatically satisfied. The only times we won't have $\mathcal{F}$ be all subsets is for technical reasons or when we talk about conditional expectation.

2.1.2. Probability axioms. So now we have a sample space S, a σ-field $\mathcal{F}$, and we need to talk about what a probability is.

Probability axioms

(1) $0 \leqslant \mathbb{P}(E) \leqslant 1$ for all events $E \in \mathcal{F}$.
(2) $\mathbb{P}(S) = 1$.
(3) If the $\{E_i\}_{i=1}^{\infty}, E_i \in \mathcal{F}$ are pairwise disjoint, $\mathbb{P}(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mathbb{P}(E_i)$.

Note that probabilities are probabilities of subsets of S, not of points of S. However, it is common to write $\mathbb{P}(x)$ for $\mathbb{P}(\{x\})$.

Intuitively, the probability of E should be the number of times E occurs in n experiments, taking a limit as n tends to infinity. This is hard to use. It is better to start with these axioms, and then to prove that the probability of E is the limit as we hoped.

Below are some easy consequences of the probability axioms.

Proposition 2.1 (Properties of probability)

(1) $\mathbb{P}(\emptyset) = 0$.
(2) If $E_1, \ldots, E_n \in \mathcal{F}$ are pairwise disjoint, then $\mathbb{P}(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} \mathbb{P}(E_i)$.
(3) $\mathbb{P}(E^c) = 1 - \mathbb{P}(E)$ for any $E \in \mathcal{F}$.
(4) If $E \subset F$, then $\mathbb{P}(E) \leqslant \mathbb{P}(F)$, for any $E, F \in \mathcal{F}$.
(5) for any $E, F \in \mathcal{F}$

$$\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F). \tag{2.1.1}$$

The last property is sometimes called the *inclusion-exclusion identity*.

Proof. To show (1), choose $E_i = \emptyset$ for each i. These are clearly disjoint, so $\mathbb{P}(\emptyset) = \mathbb{P}(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mathbb{P}(E_i) = \sum_{i=1}^{\infty} \mathbb{P}(\emptyset)$. If $\mathbb{P}(\emptyset)$ were strictly positive, then the last term would be infinity, contradicting the fact that probabilities are between 0 and 1. So the probability of $\emptyset$ must be zero.

Part (2) follows if we let $E_{n+1} = E_{n+2} = \cdots = \emptyset$. Then $\{E_i\}_{i=1}^{\infty}, E_i \in \mathcal{F}$ are still pairwise disjoint, and $\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{n} E_i$, and by (1) we have

$$\sum_{i=1}^{\infty} \mathbb{P}(E_i) = \sum_{i=1}^{n} \mathbb{P}(E_i).$$

To prove (3), use $S = E \cup E^c$. By (2), $\mathbb{P}(S) = \mathbb{P}(E) + \mathbb{P}(E^c)$. By axiom (2), $\mathbb{P}(S) = 1$, so (1) follows.

To prove (4), write $F = E \cup (F \cap E^c)$, so $\mathbb{P}(F) = \mathbb{P}(E) + \mathbb{P}(F \cap E^c) \geqslant \mathbb{P}(E)$ by (2) and axiom (1).

Similarly, to prove (5), we have $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(E^c \cap F)$ and $\mathbb{P}(F) = \mathbb{P}(E \cap F) + \mathbb{P}(E^c \cap F)$. Solving the second equation for $\mathbb{P}(E^c \cap F)$ and substituting in the first gives the desired result. □

It is common for a probability space to consist of finitely many points, all with equally likely probabilities. For example, in tossing a fair coin, we have $S = \{H, T\}$, with $\mathbb{P}(H) = \mathbb{P}(T) = \frac{1}{2}$. Similarly, in rolling a fair die, the probability space consists of $\{1, 2, 3, 4, 5, 6\}$, each point having probability $\frac{1}{6}$.

Example 2.1. What is the probability that if we roll 2 dice, the sum is 7?

Solution: There are 36 possibilities, of which 6 have a sum of 7: $(1, 6)$, $(2, 5)$, $(3, 4)$, $(4, 3)$, $(5, 2)$, $(6, 1)$. Since they are all equally likely, the probability is $\frac{6}{36} = \frac{1}{6}$.

Example 2.2. What is the probability that in a poker hand (5 cards out of 52) we get exactly 4 of a kind?

Solution: we have four suits: clubs, diamonds, hearts and spades. Each suit includes an ace, a king, queen and jack, and ranks two through ten.

For example, the probability of 4 aces and 1 king is

$$\frac{\binom{4}{4}\binom{4}{1}}{\binom{52}{5}}.$$

The probability of 4 jacks and one 3 is the same. There are 13 ways to pick the rank that we have 4 of, and then 12 ways to pick the rank we have one of, so the answer is

$$13 \cdot 12 \cdot \frac{\binom{4}{4}\binom{4}{1}}{\binom{52}{5}}.$$

Example 2.3. What is the probability that in a poker hand we get exactly 3 of a kind (and the other two cards are of different ranks)?

Solution: for example, the probability of 3 aces, 1 king and 1 queen is

$$\frac{\binom{4}{3}\binom{4}{1}\binom{4}{1}}{\binom{52}{5}}.$$

We have 13 choices for the rank we have 3 of, and $\binom{12}{2}$ choices for the other two ranks, so the answer is

$$13\binom{12}{2}\frac{\binom{4}{3}\binom{4}{1}\binom{4}{1}}{\binom{52}{5}}.$$

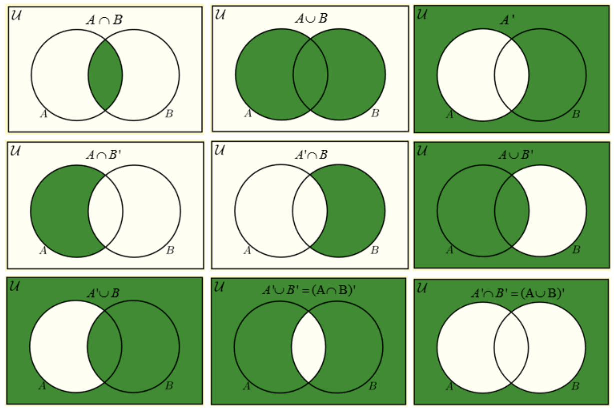
Example 2.4. In a class of 30 people, what is the probability everyone has a different birthday? (We assume each day is equally likely.)

Solution: we assume that it is not a leap year. Let the first person have a birthday on some day. The probability that the second person has a different birthday will be $\frac{364}{365}$. The probability that the third person has a different birthday from the first two people is $\frac{363}{365}$. So the answer is

$$\frac{364}{365} \cdot \frac{363}{365} \cdot \ldots \cdot \frac{336}{365}.$$

2.2. Further examples and applications

2.2.1. Sets revisited. A visual way to represent set operations is given by the Venn diagrams.



A picture of Venn diagrams from

`http://www.onlinemathlearning.com/shading-venn-diagrams.html`

Example 2.5. Roll two dice. We can describe the sample set S as *ordered pairs* of numbers $1, 2, ..., 6$, that is, S has 36 elements. Examples of events are

$$
\begin{aligned}
& E = \text{ the two dice come up equal and even } = \{(2,2), (4,4), (6,6)\}, \\
& F = \text{ the sum of the two dice is } 8 = \{(2,6), (3,5), (4,4), (5,3), (6,2)\}, \\
& E \cup F = \{(2,2), (2,6), (3,5), (4,4), (5,3), (6,2), (6,6)\}, \\
& E \cap F = \{(4,4)\}, \\
& F^c = \text{ all 31 pairs that do not include } \{(2,6), (3,5), (4,4), (5,3), (6,2)\}.
\end{aligned}
$$

Example 2.6. Let $S = [0, \infty)$ be the space of all possible ages at which someone can die. Possible events are

$A =$ person dies before reaching $30 = [0, 30)$.
$A^c = [30, \infty) =$ person dies after turning 30.
$A \cup A^c = S$,
$B =$ a person lives either less than 15 or more than 45 years $= (15, 45]$.

2.2.2. Axioms of probability revisited.

Example 2.7 (Coin tosses)**.** In this case $S = \{H, T\}$, where H stands for heads, and T stands for tails. We say that a coin is fair if we toss a coin with each side being equally likely, that is,

$$\mathbb{P}(\{H\}) = \mathbb{P}(\{T\}) = \frac{1}{2}.$$

We may write $\mathbb{P}(H) = \mathbb{P}(T) = \frac{1}{2}$. However, if the coin is biased, then still $S = \{H, T\}$ but each side can be assigned a different probability, for instance

$$\mathbb{P}(H) = \frac{2}{3}, \mathbb{P}(T) = \frac{1}{3}.$$

Example 2.8. Rolling a fair die, the probability of getting an even number is

$$\mathbb{P}(\{\text{even}\}) = \mathbb{P}(2) + \mathbb{P}(4) + \mathbb{P}(6) = \frac{1}{2}.$$

Let us see how we can use properties of probability in Proposition 2.1 to solve problems.

Example 2.9. UConn Basketball is playing Kentucky this year and from past experience the following is known:

- a home game has 0.5 chance of winning;
- an away game has 0.4 chance of winning;
- there is a 0.3 chance that UConn wins both games.

What is probability that UConn loses both games?

Solution: Let us denote by A_1 the event of a *home game win*, and by A_2 an *away game win*. Then, from the past experience we know that $\mathbb{P}(A_1) = 0.5$, $\mathbb{P}(A_2) = 0.4$ and $\mathbb{P}(A_1 \cap A_2) = 0.3$. Notice that the event *UConn loses both games* can be expressed as $A_1^c \cap A_2^c$. Thus we want to find $\mathbb{P}(A_1^c \cap A_2^c)$. Use De Morgan's laws and (3) in Proposition 2.1 we have

$$\mathbb{P}(A_1^c \cap A_2^c) = \mathbb{P}((A_1 \cup A_2)^c) = 1 - \mathbb{P}(A_1 \cup A_2).$$

The inclusion-exclusion identity (2.1.1) tells us

$$\mathbb{P}(A_1 \cup A_2) = 0.5 + 0.4 - 0.3 = 0.6,$$

and hence $\mathbb{P}(A_1^c \cap A_2^c) = 1 - 0.6 = 0.4$.

The inclusion-exclusion identity is actually true for any finite number of events. To illustrate this, we give next the formula in the case of three events.

Proposition 2.2 (Inclusion-exclusion identity)

For any three events $A, B, C \in \mathcal{F}$ in the sample state S

$$\begin{aligned}\mathbb{P}(A \cup B \cup C) = {} & \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) \\ & - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C).\end{aligned} \tag{2.2.1}$$

2.2.3. Uniform discrete distribution. If in an experiment the probability space consists of finitely many points, all with equally likely probabilities, the probability of any given event has the following simple expression.

Uniform discrete distribution

The probability of an event $E \in \mathcal{F}$ in the sample state S is given by

$$\mathbb{P}(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in } S}.$$

To show this formula rigorously we can start by considering an event E consisting of exactly one element, and use axioms of probability to see that this formula holds for such an E. Then we can represent any event E as a disjoint union of one-element events to prove the statement.

Example 2.10. A committee of 5 people is to be selected from a group of 6 men and 9 women. What is probability that it consists of 3 men and 2 women?

Solution: In this case, in counting the ways to select a group with 3 men and 2 women the order is irrelevant. We have

$$\mathbb{P}(E) = \frac{\text{the number of groups with 3 men and 2 women}}{\text{the number of groups of 5}} = \frac{\binom{6}{3}\binom{9}{2}}{\binom{15}{5}} = \frac{240}{1001}.$$

Many experiments can be modeled by considering a set of balls from which some will be withdrawn. There are two basic ways of withdrawing, namely *with* or *without* replacement.

Example 2.11. Three balls are randomly withdrawn without replacement from a bowl containing 6 white and 5 black balls. What is the probability that one ball is white and the other two are black?

Solution: this is a good example of the situation where a choice of the sample space might be different.

First proof: we model the experiment so that the order in which the balls are drawn is important. That is, we can describe the sample state S as *ordered* triples of letters W and B. Then

$$\begin{aligned} P(E) &= \frac{WBB + BWB + BBW}{11 \cdot 10 \cdot 9} \\ &= \frac{6 \cdot 5 \cdot 4 + 5 \cdot 6 \cdot 4 + 5 \cdot 4 \cdot 6}{990} = \frac{120 + 120 + 120}{990} = \frac{4}{11}. \end{aligned}$$

Second proof: we model the experiment so that the order in which the balls are drawn is *not* important. In this case

$$P(E) = \frac{(\text{one ball is white})(\text{two balls are black})}{\binom{11}{3}} = \frac{\binom{6}{1}\binom{5}{2}}{\binom{11}{3}} = \frac{4}{11}.$$

2.3. Exercises

Exercise 2.1. Consider a box that contains 3 balls: 1 red, 1 green, and 1 yellow.

(A) Consider an experiment that consists of taking 1 ball from the box, placing it back in the box, and then drawing a second ball from the box. List all possible outcomes.
(B) Repeat the experiment but now, after drawing the first ball, the second ball is drawn from the box without replacing the first. List all possible outcomes.

Exercise 2.2. Suppose that A and B are pairwise disjoint events for which $\mathbb{P}(A) = 0.2$ and $\mathbb{P}(B) = 0.4$.

(A) What is the probability that B occurs but A does not?
(B) What is the probability that neither A nor B occurs?

Exercise 2.3. Forty percent of the students at a certain college are members neither of an academic club nor of a Greek organization. Fifty percent are members of an academic club and thirty percent are members of a Greek organization. What is the probability that a randomly chosen student is

(A) member of an academic club or a Greek organization?
(B) member of an academic club and of a Greek organization?

Exercise 2.4. In a city, 60% of the households subscribe to newspaper A, 50% to newspaper B, 40% to newspaper C, 30% to A and B, 20% to B and C, and 10% to A and C. None subscribe to all three.

(A) What percentage subscribe to exactly one newspaper?(Hint: Draw a Venn diagram)
(B) What percentage subscribe to at most one newspaper?

Exercise 2.5. There are 52 cards in a standard deck of playing cards. There are 4 *suits*: hearts, spades, diamonds, and clubs ($\heartsuit\spadesuit\diamondsuit\clubsuit$). Hearts and diamonds are red while spades and clubs are black. In each suit there are 13 *ranks*: the numbers $2, 3 \ldots, 10$, the three face cards, Jack, Queen, King, and the Ace. Note that Ace is not a face card. A *poker hand* consists of five cards. Find the probability of randomly drawing the following poker hands.

(A) All 5 cards are red?
(B) Exactly two 10s and exactly three Aces?
(C) all 5 cards are either face cards or no-face cards?

Exercise 2.6. Find the probability of randomly drawing the following poker hands.

(A) A *one pair*, which consists of two cards of the same rank and three other distinct ranks. (e.g. 22Q59)
(B) A *two pair*, which consists of two cards of the same rank, two cards of another rank, and another card of yet another rank. (e.g.JJ779)
(C) A *three of a kind*, which consists of a three cards of the same rank, and two others of distinct rank (e.g. 4449K).

(D) A *flush*, which consists of all five cards of the same suit (e.g. HHHH, SSSS, DDDD, or CCCC).
(E) A *full house*, which consists of a two pair and a three of a kind (e.g. 88844). (Hint: Note that 88844 is a different hand than a 44488.)

Exercise 2.7. Suppose a standard deck of cards is modified with the additional rank of *Super King* and the additional suit of *Swords* so now each card has one of 14 ranks and one of 5 suits. What is the probability of

(A) selecting the *Super King of Swords*?
(B) getting a six card hand with exactly three pairs (two cards of one rank and two cards of another rank and two cards of yet another rank, e.g. 7,7,2,2,J,J) ?
(C) getting a six card hand which consists of three cards of the same rank, two cards of another rank, and another card of yet another rank. (e.g. 3,3,3,A,A,7)?

Exercise 2.8. A pair of fair dice is rolled. What is the probability that the first die lands on a strictly higher value than the second die?

Exercise 2.9. In a seminar attended by 8 students, what is the probability that at least two of them have birthday in the same month?

Exercise 2.10. Nine balls are randomly withdrawn without replacement from an urn that contains 10 blue, 12 red, and 15 green balls. What is the probability that

(A) 2 blue, 5 red, and 2 green balls are withdrawn?
(B) at least 2 blue balls are withdrawn?

Exercise 2.11. Suppose 4 valedictorians from different high schools are accepted to the 8 Ivy League universities. What is the probability that each of them chooses to go to a different Ivy League university?

Exercise 2.12. Two dice are thrown. Let E be the event that the sum of the dice is even, and F be the event that at least one of the dice lands on 2. Describe EF and $E \bigcup F$.

Exercise 2.13. If there are 8 people in a room, what is the probability that no two of them celebrate their birthday in the same month?

Exercise 2.14. Box I contains 3 red and 2 black balls. Box II contains 2 red and 8 black balls. A coin is tossed. If H, then a ball from box I is chosen; if T, then from from box II.

(1) What is the probability that a red ball is chosen?
(2) Suppose now the person tossing the coin does not reveal if it has turned H or T. If a red ball was chosen, what is the probability that it was box I (that is, H)?

Exercise* 2.1. Prove Proposition 2.2 by grouping $A \cup B \cup C$ as $A \cup (B \cup C)$ and using the Equation (2.1.1) for two sets.

2.4. Selected solutions

Solution to Exercise 2.1(A): Since every marble can be drawn first and every marble can be drawn second, there are $3^2 = 9$ possibilities: RR, RG, RB, GR, GG, GB, BR, BG, and BB (we let the first letter of the color of the drawn marble represent the draw).

Solution to Exercise 2.1(B): In this case, the color of the second marble cannot match the color of the rest, so there are 6 possibilities: RG, RB, GR, GB, BR, and BG.

Solution to Exercise 2.2(A): Since $A \cap B = \emptyset$, $B \subseteq A^c$ hence $\mathbb{P}(B \cap A^c) = \mathbb{P}(B) = 0.4$.

Solution to Exercise 2.2(B): By De Morgan's laws and property (3) of Proposition 2.1,

$$\mathbb{P}(A^c \cap B^c) = \mathbb{P}((A \cup B)^c) = 1 - \mathbb{P}(A \cup B) = 1 - (\mathbb{P}(A) + \mathbb{P}(B)) = 0.4.$$

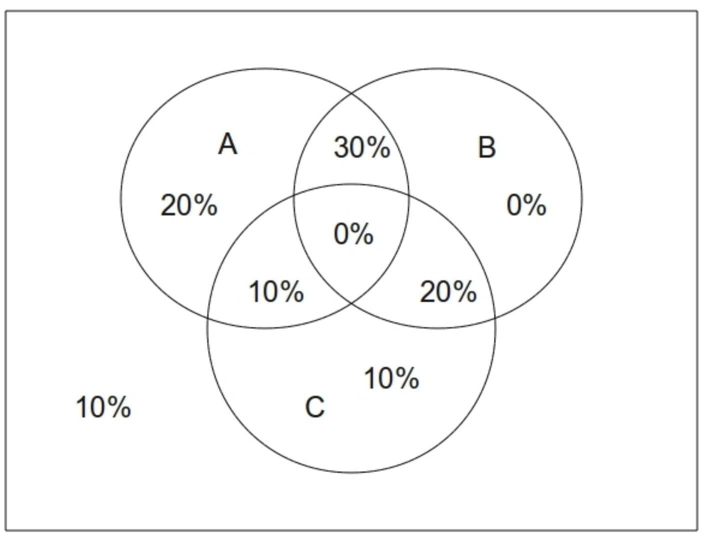
Solution to Exercise 2.3(A): $\mathbb{P}(A \cup B) = 1 - .4 = .6$

Solution to Exercise 2.3(B): Notice that

$$0.6 = \mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = 0.5 + 0.3 - \mathbb{P}(A \cap B)$$

Thus, $\mathbb{P}(A \cap B) = 0.2$.

Solution to Exercise 2.4(A): We use these percentages to produce the Venn diagram below:



This tells us that 30% of households subscribe to exactly one paper.

Solution to Exercise 2.4(B): The Venn diagram tells us that $100\% - (10\% + 20\% + 30\%) = 40\%$ of the households subscribe to at most one paper.

Solution to Exercise 2.5(A): $\dfrac{\binom{26}{5}}{\binom{52}{5}}$.

Solution to Exercise 2.5(B): $\dfrac{\binom{4}{2} \cdot \binom{4}{3}}{\binom{52}{5}}$

Solution to Exercise 2.5(C): $\frac{\binom{12}{5}}{\binom{52}{5}} + \frac{\binom{40}{5}}{\binom{52}{5}}$

Solution to Exercise 2.6(A): $\frac{13\cdot\binom{4}{2}\cdot\binom{12}{3}\cdot\binom{4}{1}\cdot\binom{4}{1}\cdot\binom{4}{1})}{\binom{52}{5}}$

Solution to Exercise 2.6(B): $\frac{\binom{13}{2}\cdot\binom{4}{2}\cdot\binom{4}{2}\cdot\binom{44}{1}}{\binom{52}{5}}$

Solution to Exercise 2.6(C): $\frac{13\cdot\binom{4}{3}\cdot\binom{12}{2}\cdot\binom{4}{1}\cdot\binom{4}{1}}{\binom{52}{5}}$

Solution to Exercise 2.6(D): $\frac{4\cdot\binom{13}{5}}{\binom{52}{5}}$

Solution to Exercise 2.6(E): $\frac{13\cdot 12\cdot\binom{4}{3}\cdot\binom{4}{2}}{\binom{52}{5}}$

Solution to Exercise 2.7(A): $\frac{1}{70}$

Solution to Exercise 2.7(B): $\frac{\binom{14}{3}\cdot\binom{5}{2}\cdot\binom{5}{2}\cdot\binom{5}{2}}{\binom{70}{6}}$

Solution to Exercise 2.7(C): $\frac{14\cdot\binom{5}{3}\cdot 13\cdot\binom{5}{2}\cdot 12\cdot\binom{5}{1}}{\binom{70}{6}}$

Solution to Exercise 2.8: we can simply list all possibilities

$$
\begin{array}{rl}
(6,1)\,,\ (6,2)\,,\ (6,3),\ (6,4)\,,\ (6,5) & \quad 5 \text{ possibilities} \\
(5,1)\,,\ (5,2),\ (5,3)\,,\ (5,4) & \quad 4 \text{ possibilities} \\
(4,1)\,,\ (4,2),\ (4,3) & \quad 3 \text{ possibilities} \\
(3,1)\,,\ (3,2) & \quad 2 \text{ possibilities} \\
(2,1) & \quad 1 \text{ possibility} \\
& = 15 \text{ possibilities in total}
\end{array}
$$

Thus the probability is $\frac{15}{36}$.

Solution to Exercise 2.9:

$$1 - \frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5}{12^8}$$

Solution to Exercise 2.10(A): $\frac{\binom{10}{2}\cdot\binom{12}{5}\cdot\binom{15}{2}}{\binom{37}{9}}$

Solution to Exercise 2.10(B):

$$1 - \frac{\binom{27}{9}}{\binom{37}{9}} - \frac{\binom{10}{1}\cdot\binom{27}{8}}{\binom{37}{9}}$$

Solution to Exercise 2.11:

$$\frac{8\cdot 7\cdot 6\cdot 5}{8^4}$$

Solution to Exercise* 2.1: to prove Proposition 2.2 we will use Equation (2.1.1) several times, as well as a distribution law for sets. First, we apply Equation (2.1.1) to two sets A and $B \cup C$ to see that

$$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A \cup (B \cup C)) = \mathbb{P}(A) + \mathbb{P}(B \cup C) - \mathbb{P}(A \cap (B \cup C)).$$

We can now apply Equation (2.1.1) to the sets B and C to see that

$$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(B \cap C) - \mathbb{P}(A \cap (B \cup C)). \tag{2.4.1}$$

Then the distribution law for sets says

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

which we can see by using the Venn diagrams. Now we can apply Equation (2.1.1) to the sets $(A \cap B)$ and $(A \cap C)$ to see that

$$\mathbb{P}(A \cap (B \cup C)) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap C) - \mathbb{P}((A \cap B) \cap (A \cap C)).$$

Finally observe that

$$(A \cap B) \cap (A \cap C) = A \cap B \cap C,$$

so

$$\mathbb{P}(A \cap (B \cup C)) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap C) - \mathbb{P}(A \cap B \cap C).$$

Use this in Equation (2.4.1) to finish the proof.