CHAPTER 2

The probability set-up

2.1. Introduction and basic theory

We will have a sample space, denoted $S$ (sometimes $\Omega$) that consists of all possible outcomes. For example, if we roll two dice, the sample space would be all possible pairs made up of the numbers one through six. An event is a subset of $S$.

Another example is to toss a coin 2 times, and let $S = \{HH, HT, TH, TT\}$; or to let $S$ be the possible orders in which 5 horses finish in a horse race; or $S$ the possible prices of some stock at closing time today; or $S = [0, \infty)$; the age at which someone dies; or $S$ the points in a circle, the possible places a dart can hit.

We use the following usual notation: $A \cup B$ is the union of $A$ and $B$ and denotes the points of $S$ that are in $A$ or $B$ or both. $A \cap B$ is the intersection of $A$ and $B$ and is the set of points that are in both $A$ and $B$. $\emptyset$ denotes the empty set. $A \subset B$ means that $A$ is contained in $B$ and $A^c$ is the complement of $A$, that is, the points in $S$ that are not in $A$. We extend the definition to have $\cup_{i=1}^n A_i$ is the union of $A_1, \cdots, A_n$, and similarly $\cap_{i=1}^n A_i$.

An exercise is to show that $(\cup_{i=1}^n A_i)^c = \cap_{i=1}^n A_i^c$ and $(\cap_{i=1}^n A_i)^c = \cup_{i=1}^n A_i^c$.

These are called DeMorgan’s laws.

There are no restrictions on $S$. The collection of events, $\mathcal{F}$, must be a $\sigma$-field, which means that it satisfies the following:

(i) $\emptyset, S$ is in $\mathcal{F}$;
(ii) if $A$ is in $\mathcal{F}$, then $A^c$ is in $\mathcal{F}$;
(iii) if $A_1, A_2, \ldots$ are in $\mathcal{F}$, then $\cup_{i=1}^\infty A_i$ and $\cap_{i=1}^\infty A_i$ are in $\mathcal{F}$.

Typically we will take $\mathcal{F}$ to be all subsets of $S$, and so (i)-(iii) are automatically satisfied. The only times we won’t have $\mathcal{F}$ be all subsets is for technical reasons or when we talk about conditional expectation.

So now we have a space $S$, a $\sigma$-field $\mathcal{F}$, and we need to talk about what a probability is. There are three axioms:

(1) $0 \leq P(E) \leq 1$ for all events $E$.
(2) $P(S) = 1$.
(3) If the $E_i$ are pairwise disjoint, $P(\cup_{i=1}^\infty E_i) = \sum_{i=1}^\infty P(E_i)$. 

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Pairwise disjoint means that $E_i \cap E_j = \emptyset$ unless $i = j$.

Note that probabilities are probabilities of subsets of $S$, not of points of $S$. However it is common to write $\mathbb{P}(x)$ for $\mathbb{P}\{\{x\}\}$.

Intuitively, the probability of $E$ should be the number of times $E$ occurs in $n$ times, taking a limit as $n$ tends to infinity. This is hard to use. It is better to start with these axioms, and then to prove that the probability of $E$ is the limit as we hoped.

There are some easy consequences of the axioms.

**Proposition 2.1:**
1. $\mathbb{P}(\emptyset) = 0$.
2. If $A_1, \ldots, A_n$ are pairwise disjoint, $\mathbb{P}(\cup_{i=1}^n A_i) = \sum_{i=1}^n \mathbb{P}(A_i)$.
3. $\mathbb{P}(E^c) = 1 - \mathbb{P}(E)$.
4. If $E \subseteq F$, then $\mathbb{P}(E) \leq \mathbb{P}(F)$.
5. $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$.

**Proof.** For (1), let $A_i = \emptyset$ for each $i$. These are clearly disjoint, so $\mathbb{P}(\emptyset) = \mathbb{P}(\bigcup_{i=1}^\infty A_i) = \sum_{i=1}^\infty \mathbb{P}(A_i) = \sum_{i=1}^\infty \mathbb{P}(\emptyset)$. If $\mathbb{P}(\emptyset)$ were positive, then the last term would be infinity, contradicting the fact that probabilities are between 0 and 1. So the probability must be zero.

The second follows if we let $A_{n+1} = A_{n+2} = \cdots = \emptyset$. We still have pairwise disjointness and $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^n A_i$, and $\sum_{i=1}^\infty \mathbb{P}(A_i) = \sum_{i=1}^n \mathbb{P}(A_i)$, using (1).

To prove (3), use $S = E \cup E^c$. By (2), $\mathbb{P}(S) = \mathbb{P}(E) + \mathbb{P}(E^c)$. By axiom (2), $\mathbb{P}(S) = 1$, so (1) follows.

To prove (4), write $F = E \cup (F \cap E^c)$, so $\mathbb{P}(F) = \mathbb{P}(E) + \mathbb{P}(F \cap E^c) \geq \mathbb{P}(E)$ by (2) and axiom (1).

Similarly, to prove (5), we have $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(E^c \cap F)$ and $\mathbb{P}(F) = \mathbb{P}(E \cap F) + \mathbb{P}(E^c \cap F)$. Solving the second equation for $\mathbb{P}(E^c \cap F)$ and substituting in the first gives the desired result.

It is very common for a probability space to consist of finitely many points, all with equally likely probabilities. For example, in tossing a fair coin, we have $S = \{H, T\}$, with $\mathbb{P}(H) = \mathbb{P}(T) = \frac{1}{2}$. Similarly, in rolling a fair die, the probability space consists of $\{1, 2, 3, 4, 5, 6\}$, each point having probability $\frac{1}{6}$.

**Example 2.1:** What is the probability that if we roll 2 dice, the sum is 7?

**Answer.** There are 36 possibilities, of which 6 have a sum of 7: $(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)$. Since they are all equally likely, the probability is $\frac{6}{36} = \frac{1}{6}$.

**Example 2.2:** What is the probability that in a poker hand (5 cards out of 52) we get exactly 4 of a kind?

**Answer.** The probability of 4 aces and 1 king is $\binom{4}{4} \binom{4}{1} / \binom{52}{5}$. The probability of 4 jacks and one 3 is the same. There are 13 ways to pick the rank that we have 4 of and then
12 ways to pick the rank we have one of, so the answer is

\[
13 \cdot 12 \frac{\binom{4}{4} \binom{4}{1}}{\binom{52}{5}}.
\]

**Example 2.3:** What is the probability that in a poker hand we get exactly 3 of a kind (and the other two cards are of different ranks)?

**Answer.** The probability of 3 aces, 1 king and 1 queen is

\[
\frac{\binom{4}{3} \binom{4}{1} \binom{4}{1}}{\binom{52}{5}}.
\]

We have 13 choices for the rank we have 3 of and \(\binom{12}{2}\) choices for the other two ranks, so the answer is

\[
13 \binom{12}{2} \frac{\binom{4}{3} \binom{4}{1} \binom{4}{1}}{\binom{52}{5}}.
\]

**Example 2.4:** In a class of 30 people, what is the probability everyone has a different birthday? (We assume each day is equally likely.)

**Answer.** Let the first person have a birthday on some day. The probability that the second person has a different birthday will be \(\frac{364}{365}\). The probability that the third person has a different birthday from the first two people is \(\frac{363}{365}\). So the answer is

\[
\frac{364}{365} \cdot \frac{363}{365} \cdots \frac{336}{365}.
\]
2.2. Further examples and applications

2.2.1. Events. An event $A$ is a subset of $S$. In this case we use the notation $A \subset S$ meaning that $A$ is a subset of $S$.

Example 2.5: Roll two dice. Examples of events are

- $E = \text{the two dice come up equal and even} = \{(2, 2), (4, 4), (6, 6)\}$,
- $F = \text{the sum of the two dice is 8} = \{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}$,
- $E \cup F = \{(2, 2), (2, 6), (3, 5), (4, 4), (5, 3), (6, 2), (6, 6)\}$,
- $E \cap F = \{(4, 4)\}$,
- $F^c = \text{all the 36 pairs that do not include}\ \{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}$.

Example 2.6: Let $S = [0, \infty)$ be the space of all possible ages at which someone can die. Possible events are

- $A = \text{person dies before reaching 30} = [0, 30)$.
- $A^c = [30, \infty) = \text{person dies after turning 30}$.
- $A \cup A^c = S$,
- $B = \text{a person lives either less than 15 or more than 45 years} = (15, 45]$.

2.2.2. Axioms of probability and their consequences.

Example 2.7: Coin tosses. Recall that if we toss a coin with each side being equally likely. Then, $S = \{H, T\}$ and

$$P(\{H\}) = P(\{T\}) = \frac{1}{2}.$$
We may write $P(H) = P(T) = \frac{1}{2}$. However, if the coin is biased, then still $S = \{H, T\}$ but each side can be assigned a different probability, for instance

$$P(H) = \frac{2}{3}, P(T) = \frac{1}{3}.$$ 

**Example 2.8:** Rolling a fair die, the probability of getting an even number is

$$P(\{\text{even}\}) = P(2) + P(4) + P(6) = \frac{1}{2}.$$ 

An important consequence of the axioms that appeared in Proposition 2.1 is the so-called *inclusion-exclusion identity* for two events $A, B \subseteq S$,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Let us see how we can combine the different consequences listed in Proposition 2.1 to solve problems.

**Example 2.9:** UConn Basketball is playing Kentucky this year and from past experience the following is known:

- Home game has .5 chance of winning
- Away game has .4 chance of winning.
- There is .3 chances that UConn wins both games.

What is the probability that UConn loses both games?

Let us write $A_1$ = “win home game”, and $A_2$ = “win away game”. Then, from past experience we know that $P(A_1) = .5$, $P(A_2) = .4$ and $P(A_1 \cap A_2) = .3$. Notice that the event “loses both games” can be expressed as $A_1^c \cap A_2^c$. Thus we want to find out $P(A_1^c \cap A_2^c)$. Simplifying as much as possible (de Morgan’s laws!) and using consequence (3) from Proposition 2.1 we have

$$P(A_1^c \cap A_2^c) = P((A_1 \cup A_2)^c) = 1 - P(A_1 \cup A_2).$$

The inclusion-exclusion identity (2.2.1) tells us

$$P(A_1 \cup A_2) = .5 + .4 - .3 = .6,$$

and hence $P(A_1^c \cap A_2^c) = 1 - .6 = .4$.

The inclusion-exclusion identity is actually true for any finite number of events. To illustrate this, we give next the formula in the case of three events.

**Proposition 2.2 (Inclusion-exclusion identity):** For any three events $A, B, C \subseteq S$,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

**Exercise 2.1:** Prove Proposition 2.2 by grouping $A \cup B \cup C$ as $A \cup (B \cup C)$ and using the formula (2.2.1) for two sets.
2.2.3. Uniform discrete distribution. If in an experiment the probability space consists of finitely many points, all with equally likely probabilities, the probability of any given event has the following simple expression.

**Proposition 2.3:** The probability of an event \( E \subseteq S \) is

\[
\mathbb{P}(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in } S}.
\]

There are different ways how to count the number of outcomes. If nothing is explicitly said we can always choose (and specify!) the way we regard our experiment.

**Example 2.10:** A committee of 5 people is to be selected from a group of 6 men and 9 women. What is probability that it consists of 3 men and 2 women?

In this case, in counting the ways to select a group with 3 men and 2 women the order is irrelevant. We have

\[
\mathbb{P}(E) = \frac{\text{groups with 3 men and 2 women}}{\text{groups of 5}} = \frac{\binom{6}{3} \binom{9}{2}}{\binom{15}{5}} = \frac{240}{1001}.
\]

Many experiments can be modeled by considering a set of balls from which some will be withdrawn. There are two basic ways of withdrawing, namely *with* or *without* replacement.

**Example 2.11:** Three balls are randomly withdrawn without replacement from a bowl containing 6 white and 5 black balls. What is the probability that one ball is white and the other two are black?

We may distinguish two cases:

a. The order in which the balls are drawn is important. Then,

\[
P(E) = \frac{WBB + BWB + BBW}{11 \cdot 10 \cdot 9} = \frac{6 \cdot 5 \cdot 4 + 5 \cdot 6 \cdot 4 + 5 \cdot 4 \cdot 6}{990} = \frac{120 + 120 + 120}{990} = \frac{4}{11}.
\]

b. The order is not important. In this case

\[
P(E) = \frac{(1 \text{ white})(2 \text{ black})}{\binom{11}{3}} = \frac{\binom{6}{1} \binom{5}{2}}{\binom{11}{3}} = \frac{4}{11}.
\]
2.3. Exercises

Exercise 2.2: Consider a box that contains 3 balls, 1 red, 1 green, and 1 yellow.

(A) Consider an experiment that consists of taking 1 ball from the box, placing it back in the box, and then drawing a second ball from the box. List all possible outcomes.
(B) Repeat the experiment but now, after drawing the first ball, the second ball is drawn from the box without replacing the first. List all possible outcomes.

Exercise 2.3: Suppose that $A$ and $B$ are pairwise disjoint events for which $\mathbb{P}(A) = 0.2$ and $\mathbb{P}(B) = 0.4$.

(A) What is the probability that $B$ occurs but $A$ does not?
(B) What is the probability that neither $A$ nor $B$ occurs?

Exercise 2.4: Forty percent of the students at a certain college are members neither of an academic club nor a Greek organization. Fifty percent are members of an academic club and thirty percent are members of a Greek organization. What is the probability that a randomly chosen student is

(A) member of an academic club or a Greek organization?
(B) member of an academic club and of a Greek organization?

Exercise 2.5: In a city, 60% of the households subscribe to newspaper $A$, 50% to newspaper $B$, 40% to newspaper $C$, 30% to $A$ and $B$, 20% to $B$ and $C$, and 10% to $A$ and $C$. None subscribe to all three.

(A) What percentage subscribe to exactly one newspaper? (Hint: Draw a Venn diagram)
(B) What percentage subscribe to at most one newspaper?

Exercise 2.6: There are 52 cards in a standard deck of playing cards. There are 4 suits: hearts, spades, diamonds, and clubs ($\heartsuit \spadesuit \diamondsuit \clubsuit$). Hearts and diamonds are red while spades and clubs are black. In each suit there are 13 ranks: the numbers 2, 3, . . . , 10, the three face cards, Jack, Queen, King, and the Ace. Note that Ace is not a face card. A poker hand consists of five cards. Find the probability of randomly drawing the following poker hands.

(A) All 5 cards are red?
(B) Exactly two 10’s and exactly three Aces?
(C) all 5 cards are either face cards or no-face cards?

Exercise 2.7: Find the probability of randomly drawing the following poker hands.

(A) A one pair, which consists of two cards of the same rank and three other distinct ranks. (e.g. 22Q59)
(B) A two pair, which consists of two cards of the same rank, two cards of another rank, and another card of yet another rank. (e.g.JJ779)
(C) A three of a kind, which consists of a three cards of the same rank, and two others of distinct rank (e.g. 4449K).
(D) A *flush*, which consists of all five cards of the same suit (e.g. HHHH, SSSS, DDDD, or CCCC).
(E) A *full house*, which consists of a two pair and a three of a kind (e.g. 88844). (Hint: Note that 88844 is a different hand than a 44488.)

**Exercise 2.8:** Suppose a standard deck of cards is modified with the additional rank of *Super King* and the additional suit of *Swords* so now each card has one of 14 ranks and one of 5 suits. What is the probability of

(A) selecting the *Super King of Swords*?
(B) getting a six card hand with exactly three pairs (two cards of one rank and two cards of another rank and two cards of yet another rank, e.g. 7,7,2,2,J,J) ?
(C) getting a six card hand which consists of three cards of the same rank, two cards of another rank, and another card of yet another rank. (e.g. 3,3,3,A,A,7)?

**Exercise 2.9:** A pair of fair dice is rolled. What is the probability that the first die lands on a strictly higher value than the second die?

**Exercise 2.10:** In a seminar attended by 8 students, what is the probability that at least two of them have birthday in the same month?

**Exercise 2.11:** Nine balls are randomly withdrawn without replacement from an urn that contains 10 blue, 12 red, and 15 green balls. What is the probability that

(A) 2 blue, 5 red, and 2 green balls are withdrawn?
(B) at least 2 blue balls are withdrawn?

**Exercise 2.12:** Suppose 4 valedictorians from different high schools are accepted to the 8 Ivy League universities. What is the probability that each of them chooses to go to a different Ivy League university?

**Exercise 2.13:** Two dice are thrown. Let $E$ be the event that the sum of the dice is even, and $F$ be the event that at least one of the dice lands on 2. Describe $EF$ and $E \cup F$.

**Exercise 2.14:** If there are 8 people in a room, what is the probability that no two of them celebrate their birthday in the same month?

**Exercise 2.15:** Box $I$ contains 3 red and 2 black balls. Box $II$ contains 2 red and 8 black balls. A coin is tossed. If H, then a ball from box $I$ is chosen; if T, then from from box $II$.

(1) What is the probability that a red ball is chosen?
(2) Suppose now the person tossing the coin does not reveal if it has turned H or T. If a red ball was chosen, what is the probability that it was box $I$ (that is, H)?
2.4. Selected solutions

Solution to Exercise 2.2(A): Since every marble can be drawn first and every marble can be drawn second, there are \(3^2 = 9\) possibilities: RR, RG, RB, GR, GG, GB, BR, BG, and BB (we let the first letter of the color of the drawn marble represent the draw).

Solution to Exercise 2.2(B): In this case, the color of the second marble cannot match the color of the rest, so there are 6 possibilities: RG, RB, GR, GB, BR, and BG.

Solution to Exercise 2.3(A): Since \(A \cap B = \emptyset\), \(B \subseteq A^c\) hence \(P(B \cap A^c) = P(B) = 0.4\).
Solution to Exercise 2.3(B): By de Morgan’s laws and property (3) of Proposition 2.1,
\[
P(A^c \cap B^c) = P((A \cup B)^c) = 1 - P(A \cup B) = 1 - (P(A) + P(B)) = 0.4.
\]

Solution to Exercise 2.4(A): \(P(A \cup B) = 1 - .4 = .6\)
Solution to Exercise 2.4(B): Notice that
\[.6 = P(A \cup B) = P(A) + P(B) - P(A \cap B) = .5 + .3 - P(A \cap B)\]
Thus, \(P(A \cap B) = .2\).

Solution to Exercise 2.5(A): We use these percentages to produce the Venn diagram below:

![Venn Diagram](image)

This tells us that 30% of households subscribe to exactly one paper.

Solution to Exercise 2.5(B): The Venn diagram tells us that \(100\% - (10\% + 20\% + 30\%) = 40\%\) of the households subscribe to at most one paper.

Solution to Exercise 2.6(A): \[
\begin{pmatrix}
26 \\
5 \\
52 \\
5
\end{pmatrix}
\]
Solution to Exercise 2.6(B): \[
\frac{\binom{4}{2} \cdot \binom{4}{3}}{\binom{52}{5}}
\]

Solution to Exercise 2.6(C): \[
\frac{\binom{12}{5} \cdot \binom{40}{5}}{\binom{52}{5} + \binom{52}{5}}
\]

Solution to Exercise 2.7(A): \[
\frac{13 \binom{4}{2} \binom{12}{3} \binom{4}{1} \binom{4}{1}}{\binom{52}{5}}
\]

Solution to Exercise 2.7(B): \[
\frac{13 \binom{4}{2} \binom{4}{2} \binom{44}{1}}{\binom{52}{5}}
\]

Solution to Exercise 2.7(C): \[
\frac{13 \binom{4}{3} \binom{12}{2} \binom{4}{1} \binom{4}{1}}{\binom{52}{5}}
\]

Solution to Exercise 2.7(D): \[
\frac{4 \binom{13}{5}}{\binom{52}{5}}
\]

Solution to Exercise 2.7(E): \[
\frac{13 \cdot 12 \binom{4}{3} \binom{4}{2}}{\binom{52}{5}}
\]

Solution to Exercise 2.8(A): \[
\frac{1}{70}
\]

Solution to Exercise 2.8(B): \[
\frac{14 \binom{5}{2} \binom{5}{2} \binom{5}{2}}{\binom{70}{6}}
\]

Solution to Exercise 2.8(C): \[
\frac{14 \binom{5}{3} 13 \binom{5}{2} 12 \binom{5}{1}}{\binom{70}{6}}
\]

Solution to Exercise 2.9: Simple inspection: we can see that the only possibilities are

\[
(6,1), (6,2), (6,3), (6,4), (6,5) \quad 5 \text{ possibilities}
\]
\[
(5,1), (5,2), (5,3), (5,4) \quad 4 \text{ possibilities}
\]
\[
(4,1), (4,2), (4,3) \quad 3 \text{ possibilities}
\]
\[
(3,1), (3,2) \quad 2 \text{ possibilities}
\]
\[
(2,1) \quad 1 \text{ possibility}
\]

\[
= 15 \text{ possibilities in total}
\]

Thus the probability is \[
\frac{15}{36}.
\]

Solution to Exercise 2.10: \[
1 - \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{12^8}
\]
Solution to Exercise 2.11(A): \[
\begin{pmatrix}
10 \\ 2 \\
12 \\ 5 \\
15 \\ 2 \\
\end{pmatrix}
\begin{pmatrix}
37 \\ 9 \\
\end{pmatrix}
\]

Solution to Exercise 2.11(B): 
\[
1 - \begin{pmatrix}
27 \\ 9 \\
\end{pmatrix} - \begin{pmatrix}
10 \\ 1 \\
\end{pmatrix} \begin{pmatrix}
27 \\ 8 \\
\end{pmatrix}
\]

Solution to Exercise 2.12: 
\[
\frac{8 \cdot 7 \cdot 6 \cdot 5}{8^4}
\]