CHAPTER 7

Continuous distributions

7.1. Basic theory

7.1.1. Definition, PDF, CDF. We start with the definition a continuous random variable.

Definition (Continuous random variables)

A random variable X is said to have a *continuous distribution* if there exists a non-negative function $f = f_X$ such that

$$\mathbb{P}(a \leqslant X \leqslant b) = \int_a^b f(x)dx$$

for every a and b. The function f is called the *density function* for X or the *PDF* for X.

More precisely, such an X is said to have an *absolutely continuous distribution*. Note that $\int_{-\infty}^{\infty} f(x)dx = \mathbb{P}(-\infty < X < \infty) = 1$. In particular, $\mathbb{P}(X = a) = \int_a^a f(x)dx = 0$ for every a.

Example 7.1. Suppose we are given that $f(x) = c/x^3$ for $x \geqslant 1$ and 0 otherwise. Since $\int_{-\infty}^{\infty} f(x)dx = 1$ and

$$c\int_{-\infty}^{\infty} f(x)dx = c\int_1^{\infty} \frac{1}{x^3}dx = \frac{c}{2},$$

we have $c = 2$.

PMF or PDF?

Probability mass function (PMF) and (probability) density function (PDF) are two names for the same notion in the case of discrete random variables. We say *PDF* or simply a *density function* for a general random variable, and we use *PMF* only for discrete random variables.

Definition (Cumulative distribution function (CDF))

The *distribution function* of X is defined as

$$F(y) = F_X(y) := \mathbb{P}(-\infty < X \leqslant y) = \int_{-\infty}^{y} f(x)dx.$$

It is also called the *cumulative distribution function* (CDF) of X.

We can define CDF for any random variable, not just continuous ones, by setting $F(y) := \mathbb{P}(X \leqslant y)$. Recall that we introduced it in Definition 5.3 for discrete random variables. In that case it is not particularly useful, although it does serve to unify discrete and continuous random variables. In the continuous case, the fundamental theorem of calculus tells us, provided f satisfies some conditions, that

$$f(y) = F'(y).$$

By analogy with the discrete case, we define the expectation of a continuous random variable.

7.1.2. Expectation, discrete approximation to continuous random variables.

Definition (Expectation)

For a continuous random variable X with the density function f we define its *expectation* by

$$\mathbb{E}X = \int_{-\infty}^{\infty} x f(x) dx$$

if this integral is absolutely convergent. In this case we call X *integrable*.

Recall that this integral is absolutely convergent if

$$\int_{-\infty}^{\infty} |x| f(x) dx < \infty.$$

In the example above,

$$\mathbb{E}X = \int_{1}^{\infty} x \frac{2}{x^3} dx = 2 \int_{1}^{\infty} x^{-2} dx = 2.$$

Later in Example 10.1 we will see that a continuous random variable with Cauchy distribution has infinite expectation.

Proposition 7.1 (Discrete approximation to continuous random variables)

Suppose X is a nonnegative continuous random variable with a finite expectation. Then there is a sequence of discrete random variables $\{X_n\}_{n=1}^{\infty}$ such that

$$\mathbb{E}X_n \xrightarrow[n\to\infty]{} \mathbb{E}X.$$

Proof. First observe that if a continuous random variable X is nonnegative, then its density $f(x) = 0$ $x < 0$. In particular, $F(y) = 0$ for $y \leqslant 0$, thought the latter is not needed for our proof. Thus for such a random variable

$$\mathbb{E}X = \int_{0}^{\infty} x f(x) dx.$$

Suppose $n \in \mathbb{N}$, then we define $X_n(\omega)$ to be $k/2^n$ if $k/2^n \leqslant X(\omega) < (k+1)/2^n$, for $k \in \mathbb{N} \cup \{0\}$. This means that we are approximating X from below by the largest multiple of 2^{-n} that is still below the value of X. Each X_n is discrete, and X_n increase to X for each $\omega \in S$.

Consider the sequence $\{\mathbb{E}X_n\}_{n=1}^{\infty}$. This sequence is an increasing sequence of positive numbers, and therefore it has a limit, possibly infinite. We want to show that it is finite and it is equal to $\mathbb{E}X$.

We have

$$\begin{aligned}\mathbb{E}X_n &= \sum_{k=1}^{\infty} \frac{k}{2^n}\mathbb{P}\left(X_n = \frac{k}{2^n}\right)\\ &= \sum_{k=1}^{\infty} \frac{k}{2^n}\mathbb{P}\left(\frac{k}{2^n} \leqslant X < \frac{k+1}{2^n}\right)\\ &= \sum_{k=1}^{\infty} \frac{k}{2^n}\int_{k/2^n}^{(k+1)/2^n} f(x)dx\\ &= \sum_{k=1}^{\infty} \int_{k/2^n}^{(k+1)/2^n} \frac{k}{2^n} f(x)dx.\end{aligned}$$

If $x \in [k/2^n, (k+1)/2^n)$, then x differs from $k/2^n$ by at most $1/2^n$, and therefore

$$\begin{aligned}0 &\leqslant \int_{k/2^n}^{(k+1)/2^n} xf(x)dx - \int_{k/2^n}^{(k+1)/2^n} \frac{k}{2^n} f(x)dx\\ &= \int_{k/2^n}^{(k+1)/2^n} \left(x - \frac{k}{2^n}\right) f(x)dx \leqslant \frac{1}{2^n}\int_{k/2^n}^{(k+1)/2^n} f(x)dx\end{aligned}$$

Note that

$$\sum_{k=1}^{\infty} \int_{k/2^n}^{(k+1)/2^n} xf(x)dx = \int_0^{\infty} xf(x)dx$$

and

$$\sum_{k=1}^{\infty} \frac{1}{2^n}\int_{k/2^n}^{(k+1)/2^n} f(x)dx = \frac{1}{2^n}\sum_{k=1}^{\infty} \int_{k/2^n}^{(k+1)/2^n} f(x)dx = \frac{1}{2^n}\int_0^{\infty} f(x)dx = \frac{1}{2^n}.$$

Therefore

$$
\begin{aligned}
0 \leqslant \mathbb{E}X - \mathbb{E}X_n &= \int_0^\infty x f(x)dx - \sum_{k=1}^\infty \int_{k/2^n}^{(k+1)/2^n} \frac{k}{2^n} f(x)dx \\
&= \sum_{k=1}^\infty \int_{k/2^n}^{(k+1)/2^n} x f(x)dx - \sum_{k=1}^\infty \int_{k/2^n}^{(k+1)/2^n} \frac{k}{2^n} f(x)dx \\
&= \sum_{k=1}^\infty \left(\int_{k/2^n}^{(k+1)/2^n} x f(x)dx - \int_{k/2^n}^{(k+1)/2^n} \frac{k}{2^n} f(x)dx \right) \\
&\leqslant \sum_{k=1}^\infty \frac{1}{2^n} \int_{k/2^n}^{(k+1)/2^n} f(x)dx = \frac{1}{2^n} \xrightarrow[n \to 0]{} 0.
\end{aligned}
$$

□

We will not prove the following, but it is an interesting exercise: if X_m is *any* sequence of discrete random variables that increase up to X, then $\lim_{m\to\infty} \mathbb{E}X_m$ will have the same value $\mathbb{E}X$.

This fact is useful to show linearity, if X and Y are positive random variables with finite expectations, then we can take X_m discrete increasing up to X and Y_m discrete increasing up to Y. Then $X_m + Y_m$ is discrete and increases up to $X + Y$, so we have

$$
\begin{aligned}
\mathbb{E}(X+Y) &= \lim_{m\to\infty} \mathbb{E}(X_m + Y_m) \\
&= \lim_{m\to\infty} \mathbb{E}X_m + \lim_{m\to\infty} \mathbb{E}Y_m = \mathbb{E}X + \mathbb{E}Y.
\end{aligned}
$$

Note that we can not easily use the approximations to X, Y and $X + Y$ we used in the previous proof to use in this argument, since $X_m + Y_m$ might not be an approximation of the same kind.

If X is not necessarily positive, we can show a similar result; we will not do the details.

Similarly to the discrete case, we have

Proposition 7.2

Suppose X is a continuous random variable with density f_X and g is a real-valued function, then

$$
\mathbb{E}g(X) = \int_{-\infty}^{\infty} g(x) f(x) dx
$$

as long as the expectation of the random variable $g(X)$ makes sense.

As in the discrete case, this allows us to define moments, and in particular the *variance*

$$
\operatorname{Var} X := \mathbb{E}[X - \mathbb{E}X]^2.
$$

As an example of these calculations, let us look at the uniform distribution.

Uniform distribution

We say that a random variable X has a *uniform distribution* on $[a, b]$ if $f_X(x) = \frac{1}{b-a}$ if $a \leqslant x \leqslant b$ and 0 otherwise.

To calculate the expectation of X

$$\begin{aligned}
\mathbb{E}X &= \int_{-\infty}^{\infty} x f_X(x) dx = \int_a^b x \frac{1}{b-a} dx \\
&= \frac{1}{b-a} \int_a^b x \, dx \\
&= \frac{1}{b-a} \left(\frac{b^2}{2} - \frac{a^2}{2} \right) = \frac{a+b}{2}.
\end{aligned}$$

This is what one would expect. To calculate the variance, we first calculate

$$\mathbb{E}X^2 = \int_{-\infty}^{\infty} x^2 f_X(x) dx = \int_a^b x^2 \frac{1}{b-a} dx = \frac{a^2 + ab + b^2}{3}.$$

We then do some algebra to obtain

$$\operatorname{Var} X = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \frac{(b-a)^2}{12}.$$

7.2. Further examples and applications

Example 7.2. Suppose X has the following p.d.f.

$$f(x) = \begin{cases} \frac{2}{x^3} & x \geqslant 1 \\ 0 & x < 1. \end{cases}$$

Find the CDF of X, that is, find $F_X(x)$. Use the CDF to find $\mathbb{P}(3 \leqslant X \leqslant 4)$.

Solution: we have $F_X(x) = 0$ if $x \leqslant 1$ and will need to compute

$$F_X(x) = \mathbb{P}(X \leqslant x) = \int_1^x \frac{2}{y^3} dy = 1 - \frac{1}{x^2}$$

when $x \geqslant 1$. We can use this formula to find the following probability

$$\begin{aligned} &\mathbb{P}(3 \leqslant X \leqslant 4) = \mathbb{P}(X \leqslant 4) - \mathbb{P}(X < 3) \\ &= F_X(4) - F_X(3) = \left(1 - \frac{1}{4^2}\right) - \left(1 - \frac{1}{3^2}\right) = \frac{7}{144}. \end{aligned}$$

Example 7.3. Suppose X has density

$$f_X(x) = \begin{cases} 2x & 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise} \end{cases}.$$

Find $\mathbb{E}X$.

Solution: we have that

$$\mathbb{E}[X] = \int x f_X(x) dx = \int_0^1 x \cdot 2x \, dx = \frac{2}{3}.$$

Example 7.4. The density of X is given by

$$f_X(x) = \begin{cases} \frac{1}{2} & \text{if } 0 \leqslant x \leqslant 2 \\ 0 & \text{otherwise} \end{cases}.$$

Find $\mathbb{E}\left[e^X\right]$.

Solution: using Proposition 7.2 with $g(x) = e^x$ we have

$$\mathbb{E}e^X = \int_0^2 e^x \cdot \frac{1}{2} dx = \frac{1}{2}\left(e^2 - 1\right).$$

Example 7.5. Suppose X has density

$$f(x) = \begin{cases} 2x & 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise} \end{cases}.$$

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Find $\operatorname{Var}(X)$.

Solution: in Example 7.3 we found $\mathbb{E}[X] = \frac{2}{3}$. Now

$$\mathbb{E}\left[X^2\right] = \int_0^1 x^2 \cdot 2x dx = 2\int_0^1 x^3 dx = \frac{1}{2}.$$

Thus

$$\operatorname{Var}(X) = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{18}.$$

Example 7.6. Suppose X has density

$$f(x) = \begin{cases} ax+b & 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise} \end{cases}.$$

and that $\mathbb{E}[X^2] = \frac{1}{6}$. Find the values of a and b.

Solution: We need to use the fact that $\int_{-\infty}^{\infty} f(x)dx = 1$ and $\mathbb{E}[X^2] = \frac{1}{6}$. The first one gives us

$$1 = \int_0^1 (ax+b)\,dx = \frac{a}{2} + b,$$

and the second one give us

$$\frac{1}{6} = \int_0^1 x^2\,(ax+b)\,dx = \frac{a}{4} + \frac{b}{3}.$$

Solving these equations gives us

$$a = -2, \text{ and } b = 2.$$

7.3. Exercises

Exercise 7.1. Let X be a random variable with probability density function

$$f(x) = \begin{cases} cx\,(5-x) & 0 \leqslant x \leqslant 5, \\ 0 & \text{otherwise.} \end{cases}$$

(A) What is the value of c?
(B) What is the cumulative distribution function of X? That is, find $F_X(x) = \mathbb{P}(X \leqslant x)$.
(C) Use your answer in part (b) to find $\mathbb{P}(2 \leqslant X \leq 3)$.
(D) What is $\mathbb{E}[X]$?
(E) What is $\operatorname{Var}(X)$?

Exercise 7.2. UConn students have designed the new U-phone. They have determined that the lifetime of a U-Phone is given by the random variable X (measured in hours), with probability density function

$$f(x) = \begin{cases} \frac{10}{x^2} & x \geqslant 10, \\ 0 & x \leq 10. \end{cases}$$

(A) Find the probability that the u-phone will last more than 20 hours.
(B) What is the cumulative distribution function of X? That is, find $F_X(x) = \mathbb{P}(X \leqslant x)$.
(C) Use part (b) to help you find $\mathbb{P}(X \geqslant 35)$?

Exercise 7.3. Suppose the random variable X has a density function

$$f(x) = \begin{cases} \frac{2}{x^2} & x > 2, \\ 0 & x \leqslant 2. \end{cases}$$

Compute $\mathbb{E}[X]$.

Exercise 7.4. An insurance company insures a large number of homes. The insured value, X, of a randomly selected home is assumed to follow a distribution with density function

$$f(x) = \begin{cases} \frac{3}{x^4} & x > 1, \\ 0 & \text{otherwise.} \end{cases}$$

Given that a randomly selected home is insured for at least 1.5, calculate the probability that it is insured for less than 2.

Exercise 7.5. The density function of X is given by

$$f(x) = \begin{cases} a + bx^2 & 0 \leqslant x \leqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

If $\mathbb{E}[X] = \frac{7}{10}$, find the values of a and b.

Exercise 7.6. Let X be a random variable with density function

$$f(x)=\begin{cases}\frac{1}{a-1} & 1<x<a,\\ 0 & \text{otherwise.}\end{cases}$$

Suppose that $\mathbb{E}[X]=6\operatorname{Var}(X)$. Find the value of a.

Exercise 7.7. Suppose you order a pizza from your favorite pizzeria at 7:00 pm, knowing that the time it takes for your pizza to be ready is uniformly distributed between 7:00 pm and 7:30 pm.

(A) What is the probability that you will have to wait longer than 10 minutes for your pizza?
(B) If at 7:15pm, the pizza has not yet arrived, what is the probability that you will have to wait at least an additional 10 minutes?

Exercise 7.8. The grade of deterioration X of a machine part has a continuous distribution on the interval $(0,10)$ with probability density function $f_X(x)$, where $f_X(x)$ is proportional to $\frac{x}{5}$ on the interval. The reparation costs of this part are modeled by a random variable Y that is given by $Y=3X^2$. Compute the expected cost of reparation of the machine part.

Exercise 7.9. A bus arrives at some (random) time uniformly distributed between $10:00$ and $10:20$, and you arrive at a bus stop at $10:05$.

(A) What is the probability that you have to wait at least 5 minutes until the bus comes?
(B) What is the probability that you have to wait at least 5 minutes, given that when you arrive today to the station the bus was not there yet (you are lucky today)?

Exercise* 7.1. For a continuous random variable X with finite first and second moments prove that

$$\begin{aligned}&\mathbb{E}(aX+b)=a\mathbb{E}X+b,\\ &\operatorname{Var}(aX+b)=a^2\operatorname{Var}X.\end{aligned}$$

for any $a,b\in\mathbb{R}$.

Exercise* 7.2. Let X be a continuous random variable with probability density function

$$f_X(x)=\frac{1}{4}xe^{-\frac{x}{2}}\mathbb{1}_{[0,\infty)}(x),$$

where the indicator function is defined as

$$\mathbb{1}_{[0,\infty)}(x)=\begin{cases}1, & 0\leqslant x<\infty;\\ 0, & \text{otherwise.}\end{cases}$$

Check that f_X is a valid probability density function, and find $\mathbb{E}(X)$ if it exists.

Exercise* 7.3. Let X be a continuous random variable with probability density function

$$f_X(x) = \frac{4\ln x}{x^3}\mathbb{1}_{[1,\infty)}(x),$$

where the indicator function is defined as

$$\mathbb{1}_{[1,\infty)}(x) = \begin{cases} 1, & 1 \leqslant x < \infty; \\ 0, & \text{otherwise.} \end{cases}$$

Check that f_X is a valid probability density function, and find $\mathbb{E}(X)$ if it exists.

7.4. Selected solutions

Solution to Exercise 7.1(A): We must have that $\int_{-\infty}^{\infty} f(x)dx = 1$, thus

$$1 = \int_0^5 cx(5-x)dx = \left[c\left(\frac{5x^2}{2} - \frac{x^3}{3}\right)\right]_0^5$$

and so we must have that $c = 6/125$.

Solution to Exercise 7.1(B): We have that

$$\begin{aligned}
F_X(x) = \mathbb{P}\left(X \leqslant x\right) &= \int_{-\infty}^{x} f(y)dy \\
&= \int_0^x \frac{6}{125} y\left(5-y\right)dx = \frac{6}{125}\left[\left(\frac{5y^2}{2} - \frac{y^3}{3}\right)\right]_0^x \\
&= \frac{6}{125}\left(\frac{5x^2}{2} - \frac{x^3}{3}\right).
\end{aligned}$$

Solution to Exercise 7.1(C): We have

$$\begin{aligned}
\mathbb{P}\left(2 \leqslant X \leqslant 3\right) &= \mathbb{P}\left(X \leqslant 3\right) - \mathbb{P}\left(X < 2\right) \\
&= \frac{6}{125}\left(\frac{5\cdot 3^2}{2} - \frac{3^3}{3}\right) - \frac{6}{125}\left(\frac{5\cdot 2^2}{2} - \frac{2^3}{3}\right) \\
&= 0.296.
\end{aligned}$$

Solution to Exercise 7.1(D): we have

$$\begin{aligned}
\mathbb{E}\left[X\right] &= \int_{-\infty}^{\infty} x f_X(x)dx = \int_0^5 x \cdot \frac{6}{125} x(5-x)dx \\
&= 2.5.
\end{aligned}$$

Solution to Exercise 7.1(E): We need to first compute

$$\begin{aligned}
\mathbb{E}\left[X^2\right] &= \int_{-\infty}^{\infty} x^2 f_X(x)dx = \int_0^5 x^2 \cdot \frac{6}{125} x(5-x)dx \\
&= 7.5.
\end{aligned}$$

Then

$$\operatorname{Var}(X) = \mathbb{E}\left[X^2\right] - \left(\mathbb{E}\left[X\right]\right)^2 = 7.5 - (2.5)^2 = 1.25.$$

Solution to Exercise 7.2(A): We have

$$\int_{20}^{\infty} \frac{10}{x^2} dx = \frac{1}{2}.$$

Solution to Exercise 7.2(B): We have

$$F(x) = \mathbb{P}(X \leqslant x) = \int_{10}^{x} \frac{10}{y^2} dy = 1 - \frac{10}{x}$$

for $x > 10$, and $F(x) = 0$ for $x < 10$.

Solution to Exercise 7.2(C): We have

$$\begin{aligned}\mathbb{P}(X \geqslant 35) &= 1 - \mathbb{P}(X < 35) = 1 - F_X(35) \\ &= 1 - \left(1 - \frac{10}{35}\right) = \frac{10}{35}.\end{aligned}$$

Solution to Exercise 7.3: $+\infty$

Solution to Exercise 7.4: $\frac{37}{64}$.

Solution to Exercise 7.5: we need to use the fact that $\int_{-\infty}^{\infty} f(x)dx = 1$ and $\mathbb{E}[X] = \frac{7}{10}$. The first one gives us

$$1 = \int_0^1 \left(a + bx^2\right) dx = a + \frac{b}{3}$$

and the second one gives

$$\frac{7}{10} = \int_0^1 x\left(a + bx^2\right) dx = \frac{a}{2} + \frac{b}{4}.$$

Solving these equations gives

$$a = \frac{1}{5}, \text{ and } b = \frac{12}{5}.$$

Solution to Exercise 7.6: Note that

$$\mathbb{E}X = \int_1^a \frac{x}{a-1} dx = \frac{1}{2}a + \frac{1}{2}.$$

Also

$$\operatorname{Var}(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2$$

then we need

$$\mathbb{E}X^2 = \int_1^a \frac{x^2}{a-1} dx = \frac{1}{3}a^2 + \frac{1}{3}a + \frac{1}{3}.$$

Then

$$\begin{aligned}\operatorname{Var}(X) &= \left(\frac{1}{3}a^2 + \frac{1}{3}a + \frac{1}{3}\right) - \left(\frac{1}{2}a + \frac{1}{2}\right)^2 \\ &= \frac{1}{12}a^2 - \frac{1}{6}a + \frac{1}{12}.\end{aligned}$$

Then, using $\mathbb{E}[X] = 6\operatorname{Var}(X)$, we simplify and get $\frac{1}{2}a^2 - \frac{3}{2}a = 0$, which gives us $a = 3$.

Another way to solve this problem is to note that, for the uniform distribution on $[a, b]$, the mean is $\frac{a+b}{2}$ and the variance is $\frac{(a-b)^2}{12}$. This gives us an equation $6\frac{(a-1)^2}{12} = \frac{a+1}{2}$. Hence $(a-1)^2 = a + 1$, which implies $a = 3$.

Solution to Exercise 7.7(A): Note that X is uniformly distributed over $(0, 30)$. Then

$$\mathbb{P}(X > 10) = \frac{2}{3}.$$

Solution to Exercise 7.7(B): Note that X is uniformly distributed over $(0, 30)$. Then

$$\mathbb{P}(X > 25 \mid X > 15) = \frac{\mathbb{P}(X > 25)}{\mathbb{P}(X > 15)} = \frac{5/30}{15/30} = 1/3.$$

Solution to Exercise 7.8: First of all we need to find the PDF of X. So far we know that

$$f(x) = \begin{cases} \frac{cx}{5} & 0 \leqslant x \leqslant 10, \\ 0 & \text{otherwise.} \end{cases}$$

Since

$$\int_0^{10} c\frac{x}{5} dx = 10c,$$

we have $c = \frac{1}{10}$. Now, applying Proposition 7.2 we get

$$\mathbb{E}Y = \int_0^{10} \frac{3}{50} x^3 dx = 150.$$

Solution to Exercise 7.9(A): The probability that you have to wait at least 5 minutes until the bus comes is $\frac{1}{2}$. Note that with probability $\frac{1}{4}$ you have to wait less than 5 minutes, and with probability $\frac{1}{4}$ you already missed the bus.

Solution to Exercise 7.9(B): The conditional probability is $\frac{2}{3}$.