CHAPTER 16

* Applications in Insurance and Actuarial Science

16.1 Introduction

Suppose that for a period, you face the risk of losing something that is unpredictable, and denote this potential loss by a random variable X. This loss may be the result of damages or injuries from (a) an automobile accident, (b) fire, theft, storm or hurricane at home, (c) premature death of head of the household, or (d) hospitalization due to an illness. Insurance allows you to exchange facing this potential loss for a fixed price or premium. It is one of the responsibilities of an actuary to assess the fair price given the nature of the risk. Actuarial science is a discipline that deals with events that are uncertain and their economic consequences; the concepts of probability and statistics provide for indispensable tools in measuring and managing these uncertainties.

16.2 The Pareto distribution

The Pareto distribution is commonly used to describe and model insurance losses. One reason is its flexibility to handle positive skewness or long distribution tails. It is possible for insurance losses to become extremely large, although such may be considered rare event. While there are several versions of the Pareto family of distributions, we consider the cumulative distribution function of X that follows a Type II Pareto:

$$F_X(x) = 1 - \left(\frac{\theta}{x+\theta}\right)^{\alpha}, \quad \text{for } x > 0, \tag{16.0.1}$$

where $\alpha > 0$ is the shape or tail parameter and $\theta > 0$ is the scale parameter. If X follows such distribution, we write $X \sim \text{Pareto}(\alpha, \theta)$.

Example 16.2.1 Consider loss X with $\text{Pareto}(3, 100)$ distribution.

(a) Calculate the probability that loss exceeds 50.
(b) Given that loss exceeds 50, calculate the probability that loss exceeds 75.

Solution: From (16.0.1), we find

$$\mathbb{P}(X > 50) = 1 - F_X(50) = \left(\frac{100}{50+100}\right)^3 = \left(\frac{2}{3}\right)^3 = \frac{8}{27} = 0.2963.$$

and

$$\mathbb{P}(X > 75 | X > 50) = \frac{\mathbb{P}(X > 75)}{\mathbb{P}(X > 50)} = \left(\frac{150}{175}\right)^3 = \left(\frac{6}{7}\right)^3 = 0.6297.$$

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By taking the derivative of (16.0.1), it can be shown that the probability density function of X is given by

$$f_X(x) = \frac{\alpha}{\theta}\left(\frac{\theta}{x+\theta}\right)^{\alpha+1}, \quad \text{for } x > 0. \tag{16.0.2}$$

Figure 16.0.1 depicts shapes of the density plot with varying parameter values of α and θ.

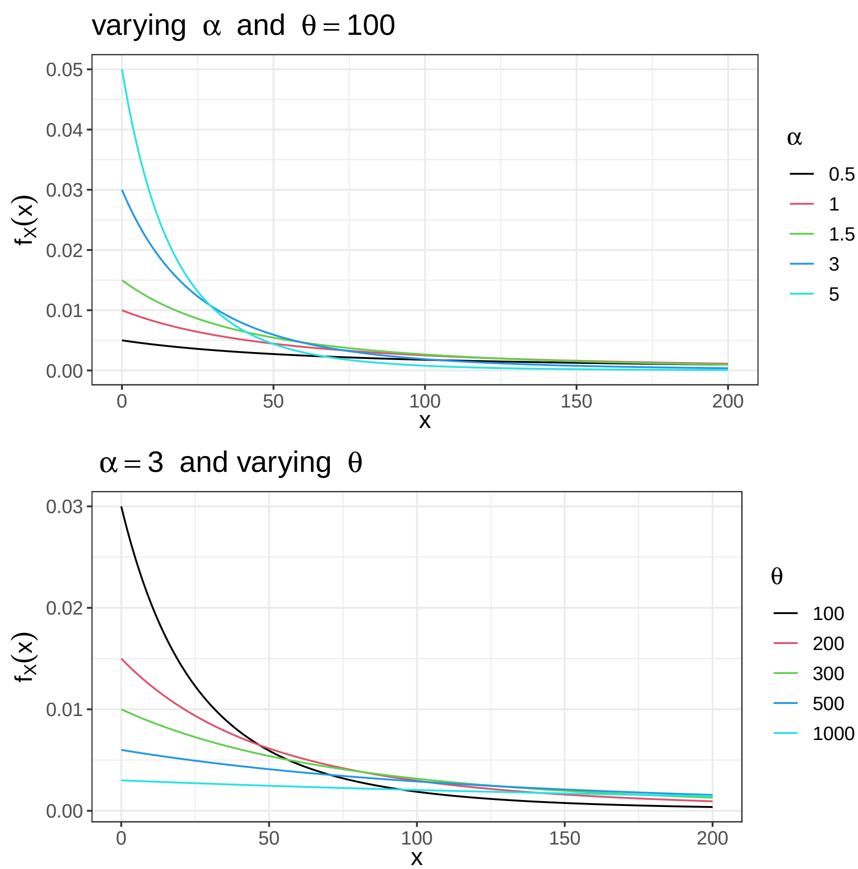

Figure 16.0.1. Density plots of the Pareto distribution.

The mean of the distribution can be shown as

$$\mathbb{E}(X) = \frac{\theta}{\alpha - 1}. \tag{16.0.3}$$

This mean exists provided $\alpha > 1$. For the variance, it can be shown that it has the expression

$$\mathrm{Var}(X) = \frac{\alpha\theta^2}{(\alpha-1)(\alpha-2)}. \tag{16.0.4}$$

This variance exists provided $\alpha > 2$, is infinite for $1 < \alpha \leq 2$, and is otherwise undefined.

16.3 Insurance coverage modifications

In some instances, the potential loss that you face may be huge and unlimited. In this case, the cost of the insurance coverage may be burdensome. There are possible modifications to your insurance coverage so that the burden may be reduced. We introduce three possible modifications: (a) deductibles, (b) limits or caps, and (c) coinsurance.

These coverage modifications are a form of loss sharing between you, who is called the policyholder or insured, and the insurance company, which is also called the insurer. The effect is a reduced premium to the policyholder, and at the same time, because the policyholder shares in the loss, there is a perceived notion that this may alter the behavior of the policyholder. For instance, in the case of automobile insurance, the policyholder may be more careful about his or her driving behavior. Note that it is also possible to have an insurance coverage which is a combination of these three modifications.

16.3.1 Deductibles

In an *excess-of-loss* insurance contract, the insurance company agrees to reimburse the policyholder for losses beyond a pre-specified amount d. This amount d is referred to as the deductible of the contract. Given the loss is X, this amount is then shared between the policyholder, who is responsible for the first d amount, and the insurance company, which pays the excess if any. Thus, the policyholder is responsible for $\min(X, d)$ and the insurance company pays the excess, which is then equal to

$$X_I = X - \min(X, d) = \begin{cases} 0, & \text{if } X \leq d \\ X - d, & \text{if } X > d \end{cases}. \tag{16.0.5}$$

In general, we keep this notation, X_I, to denote the portion of X that the insurance company agrees to pay. The expected value of this can be expressed as

$$\mathbb{E}(X_I) = \mathbb{E}(X) - \mathbb{E}[\min(X, d)], \tag{16.0.6}$$

where $\mathbb{E}[\min(X, d)]$ is sometimes called the *limited expected value.* For any non-negative random variable X, it can be shown that

$$\mathbb{E}[\min(X, d)] = \int_0^d [1 - F_X(x)]\, dx. \tag{16.0.7}$$

This result can be proved as follows. Starting with

$$\begin{aligned} \mathbb{E}[\min(X, d)] &= \int_0^d x f_X(x) dx + \int_d^\infty d \cdot f_X(x) dx \\ &= \int_0^d x f_X(x) dx + d\,[1 - F_X(d)]\,, \end{aligned}$$

and applying integration by parts with $u = x$ and $dv = -f_X(x)dx$ so that $v = 1 - F_X(x)$, we have

$$\begin{aligned} \mathbb{E}[\min(X, d)] &= -\,x\,[1 - F_X(x)]\Big|_0^d + \int_0^d [1 - F_X(x)]\, dx + d\,[1 - F_X(d)]\,, \\ &= -d\,[1 - F_X(d)] + \int_0^d [1 - F_X(x)]\, dx + d\,[1 - F_X(d)] \\ &= \int_0^d [1 - F_X(x)]\, dx, \end{aligned}$$

which gives the desired result.

Example 16.3.1 Show that the limited expected value, with a deductible d, for a Pareto(α,θ) distribution has the following expression

$$\mathbb{E}[\min(X,d)] = \frac{\theta}{\alpha-1}\left[1-\left(\frac{\theta}{d+\theta}\right)^{\alpha-1}\right], \tag{16.0.8}$$

provided $\alpha \neq 1$.

Solution: From (16.0.1), we find

$$1-F_X(x) = \left(1+\frac{x}{\theta}\right)^{-\alpha}.$$

Applying the substitution $u = 1 + x/\theta$ so that $du = (1/\theta)dx$, we get

$$\mathbb{E}[\min(X,d)] = \int_1^{1+x/\theta} \theta u^{-\alpha} du.$$

Evaluating this integral, we get the result in (16.0.8).

Example 16.3.2 An insurance company offers an *excess-of-loss* contract against loss X. Assume that X has a Pareto distribution with mean 100 and variance 200000/3. You are given that $\alpha > 3$. The insurer's expected payment for this loss is 80. Calculate the deductible amount d.

Solution: First, we find the parameters of the distribution. From (16.0.3) and (16.0.4), we have two equations in two unknowns:

$$\frac{\theta}{\alpha-1} = 100 \quad \text{and} \quad \frac{\alpha\theta^2}{(\alpha-1)(\alpha-2)} = 200000/3$$

This leads us to $\theta = 100(\alpha-1)$ which results to a quadratic equation in α: $3\alpha^2 - 23\alpha + 40 = 0$. There are two possible solutions for α: either $\alpha = 5$ or $\alpha = 8/3$. Since we are given $\alpha > 3$, we use $\alpha = 5$ so that $\theta = 400$. Calculating the insurer's expected payment, we have from (16.0.6) and (16.0.8) the following:

$$\mathbb{E}(X_I) = 100\left(\frac{400}{400+d}\right)^4 = 80.$$

Solving for the deductible, we have $d = 22.95$.

16.3.2 Policy limits

An insurance contract with a policy limit, or cap, L is called a *limited* contract. For this contract, the insurance company is responsible for payment of X provided the loss does not exceed L, and if it does, the maximum payment it will pay is L. We then have

$$X_I = \min(X,L) = \begin{cases} X, & \text{if } X \leq L \\ L, & \text{if } X > L \end{cases} \tag{16.0.9}$$

Example 16.3.3 An insurance company offers a *limited* contract against loss X that follows a Pareto distribution. The parameters are assumed to be $\alpha = 3.5$ and $\theta = 250$. Calculate the company's expected payment for this loss if the limit $L = 5000$.

Solution: From (16.0.8), we have

$$\mathbb{E}(X_I) = \mathbb{E}[\min(X, 5000)] = \frac{250}{2.5}\left[1 - \left(\frac{250}{5000 + 250}\right)^{2.5}\right] = 99.95.$$

Example 16.3.4 Suppose loss X follows an exponential distribution with mean μ. An insurance company offers a *limited* contract against loss X. If the policy limit is $L = 500$, the company's expected payment is 216.17. If the policy limit is $L = 1000$, the company's expected payment is 245.42. Calculate the mean parameter μ.

Solution: For an exponential with mean parameter μ, its density can be expressed as $f_X(x) = (1/\mu)e^{(-x/\mu)}$, for $x > 0$. This gives a distribution function equal to $F_X(x) = 1 - e^{(-x/\mu)}$. From (16.0.7), we can show that

$$\mathbb{E}[\min(X, L)] = \mu\left[1 - e^{(-L/\mu)}\right].$$

From the given, this leads us to the following two equations: $216.17 = \mu\left[1 - e^{(-500/\mu)}\right]$ and $245.42 = \mu\left[1 - e^{(-1000/\mu)}\right]$. If we let $k = e^{(500/\mu)}$ and divide the two equations, we get the quadratic equation:

$$\frac{216.17}{245.42} = \frac{1-k}{1-k^2} \implies 1 + k = \frac{245.42}{216.17} \implies k == 0.1353102,$$

provided $k \neq 1$. When $k = 1$, μ is undefined, and therefore, $k = 0.1353102$. This gives $\mu = 249.9768 \approx 250$.

16.3.3 Coinsurance

In an insurance contract with coinsurance, the insurance company agrees to pay a fixed and pre-specified proportion of k for each loss. This proportion k must be between 0 and 100%. The company's payment for each loss is

$$X_I = kX, \tag{16.0.10}$$

where $0 < k \leq 1$. Therefore, the expected value of this payment is just a fixed proportion k of the average of the loss.

Example 16.3.5 An insurance company offers a contract with *coinsurance* of 80%. Assume loss X follows a Pareto distribution with $\theta = 400$ and $\mathbb{P}(X > 100) = 0.5120$.

(a) Calculate the company's expected payment for each loss.
(b) Suppose the company agrees to replace the contract with an *excess-of-loss* coverage. Find the deductible d so that the company has the same expected payment.

Solution: From (16.0.1), we have

$$\mathbb{P}(X > 100) = \left(\frac{400}{500}\right)^{\alpha} = 0.5120.$$

This gives $\alpha = \ln(0.5120)/\ln(0.80) = 3$. Thus, the company's expected payment for each loss is $0.80 \times (400/2) = 160$. For part (b), we use equations (16.0.6) and (16.0.8) to solve for d. We have, with deductible d, the following

$$\mathbb{E}(X_I) = \frac{400}{2}\left(\frac{400}{d+400}\right)^2 = 160.$$

Solving for the deductible, we get $d = (400/(.8)^{0.5}) - 400 = 47.2136$.

16.3.4 Combination

It is not uncommon to find insurance contracts that combine the three different coverage modifications described in the previous sections. In particular, consider the general situation where we have a combination of a deductible d, policy limit L, and coinsurance k. In this case, the insurance contract will pay for a proportion k of the loss X, in excess of the deductible d, subject to the policy limit of L. The company's payment for each loss can be written as

$$X_I = \begin{cases} 0, & \text{if } X \le d, \\ k(X-d), & \text{if } d < X \le L, \\ k(L-d), & \text{if } X > L, \end{cases} \tag{16.0.11}$$

where $0 < k \le 1$. This payment can be also be expressed as

$$X_I = k\left[\min(X,L) - \min(X,d)\right], \tag{16.0.12}$$

where its expectation can be evaluated using $\mathbb{E}(X_I) = k\left[\mathbb{E}(\min(X,L)) - \mathbb{E}(\min(X,d))\right]$.

Example 16.3.6 An insurer offers a proportional *excess-of-loss* medical insurance policy. The policy is subject to a coinsurance of 90% with a deductible of 25, but has no policy limit. Assume medical loss X follows an exponential distribution with mean 800. Calculate the expected reimbursement the policyholder will receive in the event of a loss.

Solution: Since there is no policy limit, the reimbursement can be expressed as

$$X_I = k\left[X - \min(X,d)\right]$$

For X that has an exponential distribution with mean μ, we have

$$\mathbb{E}[\min(X,d)] = \mu\left[1 - e^{(-d/\mu)}\right]$$

This leads us to the average reimbursement as $\mathbb{E}(X_I) = k\mu e^{-d/\mu} = 0.90 * 800 * e^{-25/800} = 697.85$. In effect, the policyholder can expect to be reimbursed $(697.85/800) = 87\%$ of each loss.

16.4 Loss frequency

In practice, the insurance company pays for damages or injuries to insured only if the specified insured event happens. For example, in an automobile insurance, the insurance company will pay only in the event of an accident. It is therefore important to consider also the probability that an accident occurs. This refers to the loss frequency.

For simplicity, start with the Bernoulli random variable I, which indicates an accident, or some other insured event, occurs. Assume that I follows a Bernoulli distribution with p denoting the probability the event happens, i.e., $\mathbb{P}(I = 1) = p$. If the insured event happens, the amount of loss is X, typically a continuous random variable. This amount of loss is referred to as the loss severity. Ignoring any possible coverage modifications, the insurance company will pay 0, if the event does not happen, and X, if the event happens. In effect, the insurance claim can be written as the random variable

$$Y = X \cdot I = \begin{cases} 0, & \text{if } I = 0 \quad \text{(event does not happen)} \\ X, & \text{if } I = 1 \quad \text{(event happens)} \end{cases} \tag{16.0.13}$$

The random variable Y is a mixed random variable and will be called the *insurance claim*. It has a probability mass at 0 and a continuous distribution for positive values. By conditioning on the Bernoulli I, it can be shown that the cumulative distribution function of Y has the expression

$$F_Y(y) = \begin{cases} 1 - p, & \text{if } y = 0 \\ (1-p) + pF_X(y), & \text{if } y > 0 \end{cases} \tag{16.0.14}$$

This result can be shown by using the law of total probability.

Denote the mean of X by μ_X and its standard deviation by $\sigma_X > 0$. It can be shown that the expected value of Y has the expression

$$\mathbb{E}(Y) = p\mu_X \tag{16.0.15}$$

and the variance is

$$\mathrm{Var}(Y) = p(1-p)\mu_X^2 + p\sigma_X^2. \tag{16.0.16}$$

Example 16.4.1 Consider an insurance contract that covers an event with probability 0.25 that it happens, and the loss severity X has a Pareto$(3, 1000)$ distribution.

(a) Calculate the probability that the insurance contract will pay an amount less than or equal to 500.
(b) Calculate the probability that the insurance contract will pay an amount larger than 750.
(c) Calculate the expected value and variance of the insurance claim.

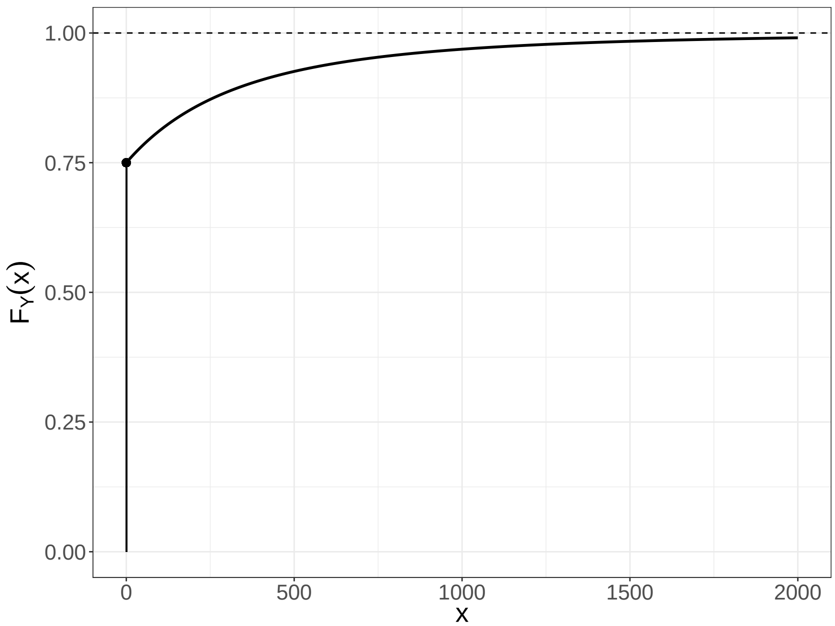

Figure 16.0.2. Cumulative distribution function of insurance claim random variable Y.

Solution: From (16.0.14), we have

$$\begin{aligned}\mathbb{P}(Y \le 500) &= \mathbb{P}(Y = 0) + \mathbb{P}(0 < Y \le 500) \\ &= (1 - 0.25) + 0.25 F_X(500) \\ &= 0.75 + 0.25\left[1 - \left(\frac{1000}{500 + 1000}\right)^3\right] = 0.9259.\end{aligned}$$

For part (b), we use the complement of the cumulative distribution function:

$$\begin{aligned}\mathbb{P}(Y > 600) &= 1 - [(1 - 0.25) + 0.25 F_X(500)] \\ &= 0.25 - 0.25\left[1 - \left(\frac{5}{8}\right)^3\right] = 0.0610.\end{aligned}$$

For the claim severity X, the mean is $\mu_X = 500$ and the variance is $\sigma_X^2 = \frac{3}{2}(1000^2)$. Therefore, from (16.0.15) and (16.0.16), we have

$$\mathbb{E}(Y) = 0.25(500) = 125$$

and

$$\mathrm{Var}(Y) = 0.25(0.75)(500^2) + 0.25 \cdot \frac{3}{2}(1000^2) = 421,875.$$

Figure 16.0.2 shows the graph of the cumulative distribution function of Y using the parameters of this example.

Example 16.4.2 An insurance company models claims using $Y = X \cdot I$, where I is the indicator of the event happening with probability 0.20. The amount of loss X has a discrete distribution given as follows:

x	20	100	500	1000
$\mathbb{P}(X = x)$	0.30	0.50	0.15	0.05

(a) Calculate the probability that the insurance claim will be below 150.
(b) Calculate the expected value of the insurance claim.
(c) Calculate the standard deviation of the insurance claim.

Solution: We have

$$\begin{aligned}\mathbb{P}(Y \leq 150) &= \mathbb{P}(Y = 0) + 0.20 * \mathbb{P}(X \leq 150) \\ &= (1 - 0.20) + 0.20(0.30 + 0.50) = 0.96\end{aligned}$$

For part (b), first we find $\mathbb{E}(X) = 182.50$ so that $\mathbb{E}(Y) = 0.20 * 182.50 = 36.5$. For part (c), we find $\sigma_X^2 = 59381.25$ so that

$$\mathrm{Var}(Y) = 0.20(0.80) * 182.5^2 + 0.20 * 59381.25 = 17205.25.$$

Therefore, the standard deviation is $\sqrt{17205.25} = 131.17$.

16.5 The concept of risk pooling

Insurance is based on the idea of pooling several individuals willing to exchange their risks. Consider the general situation where there are n individuals in the pool. Assume that each individual faces the same loss distribution but the eventual losses from each of them will be denoted by $Y_1, Y_2, \ldots, Y_n$. In addition, assume these individual losses are independent. The total loss arising from this pool is the sum of all these individual losses, $S_n = Y_1 + Y_2 + \cdots + Y_n$. To support funding this total loss, each individual agrees to contribute an equal amount of

$$P_n = \frac{1}{n} S_n = \frac{1}{n} \sum_{k=1}^{n} Y_k, \tag{16.0.17}$$

which is the average loss.

Notice that the contribution above is still a random loss. Indeed in the absence of risk pooling where there is a single individual in the pool, that individual is responsible for his or her own loss. However, when there is sufficiently large enough number of individuals, this average contribution becomes more predictable as shown below.

Assume that the expected value and variance of each loss are given, respectively, by

$$\mathbb{E}(Y_k) = \mu \quad \text{and} \quad \mathrm{Var}(Y_k) = \sigma^2.$$

For the pool of n individuals, the mean of the average loss is then

$$\mathbb{E}(P_n) = \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}(Y_k) = \mu,$$

which is exactly equal to the mean of each individual loss. However, what is interesting is the variability is reduced as shown below:

$$\mathrm{Var}(P_n) = \frac{1}{n}\sum_{k=1}^{n} \mathrm{Var}(Y_k) = \frac{1}{n}\sigma^2.$$

The variance is further reduced as the number of individuals in the pool increases. As discussed in Chapter 14, one version of the law of large numbers is the SLLN (Strong Law of Large Numbers), that accordingly, $P_n \to \mu$ as $n \to \infty$. In words, the unpredictable loss for a single individual becomes much more predictable. This is sometimes referred to as the basic law of insurance. The origins of insurance can be traced back from the idea of pooling the contributions of several for the indemnification of losses against the misfortunes of the few.

In principle, the insurance company acts as a third party that formally makes this arrangement. The company forms a group of such homogeneous and independent risks, and is responsible for collecting the individual contributions, called the premium, as well as disbursing payments when losses occur. There are additional responsibilities of the insurance company such as ensuring enough capital to cover future claims, however, such are beyond the scope here.

Figure 16.0.3 depicts the basic law of insurance. In all situations, the mean of the average loss are the same for all cases. The variability, however, is reduced with increasing number of policyholders. As you probably suspect from these graphs, the average loss is about 1000, which is sometimes referred to as the actuarially fair premium. However, please bear in mind that there are conditions for the basic law of insurance to effectively work:

- Losses must be unpredictable.
- The individual risks must be independent.
- The individuals insured must be considered homogeneous, that is, they share common risk characteristics.
- The number of individuals in the pool must be sufficiently large.

Example 16.5.1 A company insures a group of 200 homogeneous and independent policyholders against an event that happens with probability 0.18. If the event happens and therefore a loss occurs, the amount of loss X has an exponential distribution with mean 1250.

(a) Calculate the mean of the total loss arising from the group.
(b) Calculate the variance of the total loss arising from the group.
(c) Using the Central Limit Theorem, estimate the probability that the total loss will exceed 60,000.

Solution: The total loss can be written as $S_{200} = \sum_{k=1}^{200} Y_k = \sum_{k=1}^{200} I_k X_k$, where I_k is the Bernoulli random variable for the event to happen and X_k is the loss, given an event occurs. For each k, we find that

$$\mathbb{E}(Y_k) = p\mu_X = 0.18 * 1250 = 225$$

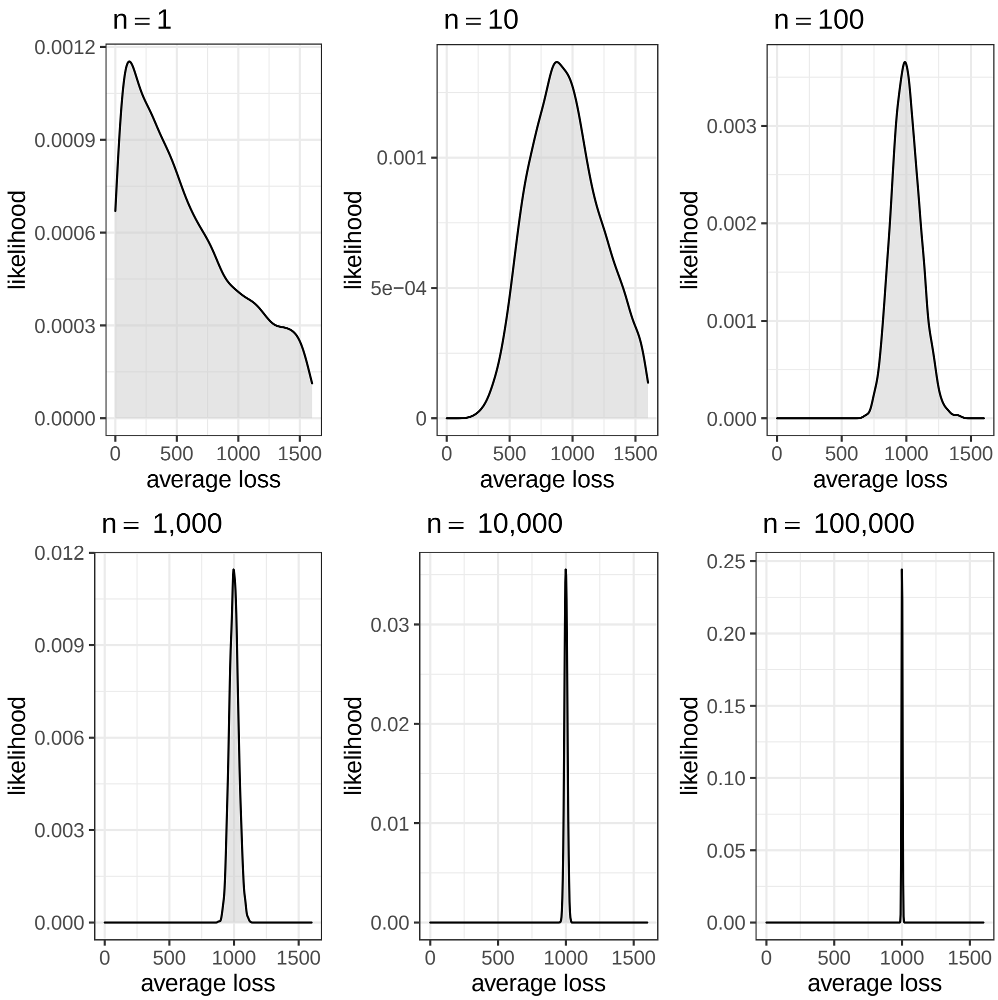

Figure 16.0.3. Distributions of the average loss as the number of policyholders increases.

and

$$\mathrm{Var}(Y_k) = p(1-p)\mu_X^2 + p\sigma_X^2 = 0.18 * .82 * 1250^2 + 0.18 * (1250^2) = 511875$$

The mean and variance of the total variance are, respectively, given by

$$\mathbb{E}(S_{200}) = 200 * 225 = 45000 \quad \text{and} \quad \mathrm{Var}(S_{200}) = 200 * 511875 = 102375000$$

The probability that the total loss will exceed 60,000 can be estimated as follows:

$$\mathbb{P}(S_{200} > 60000) \approx \mathbb{P}(Z > (60000 - 45000)/\sqrt{102375000}) = \mathbb{P}(Z > 1.48),$$

where Z denotes a standard normal random variable. From Table 1 on page 114, we get 0.06944.

16.6 Exercises

Exercise 16.1. Prove equations (16.0.3) and (16.0.4).

Exercise 16.2. Suppose insurance loss X has a Pareto(α, θ) distribution. You are given $\theta = 225$ and $\mathbb{P}(X \leq 125) = 0.7621$. Calculate the probability that loss will exceed 200, given it exceeds 125.

Exercise 16.3. Find an expression for the limited expected value, with a deductible d, for a Pareto(α, θ) distribution when $\alpha = 1$.

Exercise 16.4. An insurance company pays for a random loss X subject to a deductible amount of d, where $0 < d < 10$. The loss amount is modeled as a continuous random variable with density function

$$f_X(x) = \frac{1}{50}x, \quad \text{for } 0 < x \leq 10.$$

The probability that given the company will make a payment, it will pay less than or equal to 5 is 0.4092. Calculate d.

Exercise 16.5 An insurer offers a *limited* contract against loss X that follows an exponential distribution with mean 500. The limit of the contract is $L = 1000$. Calculate the probability that the loss will not reach the limit L.

Exercise 16.6 A company insures a loss X with a coinsurance of 90%. Loss X follows a distribution that is uniform on $(0, u)$, for $u > 0$. The expected payment for each loss for the company is 900. Calculate the expected amount of each loss the buyer of this policy is responsible for.

Exercise 16.7 An insurer offers a comprehensive medical insurance contract with a deductible of 25, subject to a coinsurance of 90% and a policy limit of 2000. Medical loss X follows an exponential distribution with mean 800.

(a) Calculate the expected reimbursement the policyholder will receive in the event of a loss.
(b) The insurer wants to reduce this expected reimbursement to 575 by adjusting only the level of deductible d. What amount d is needed to achieve this?

Exercise 16.8 Consider an insurance contract that covers an event with probability 0.10 that it happens, and the loss severity X has a Pareto$(2.5, 400)$ distribution.

(a) Calculate the probability that the insurance claim will exceed 150.
(b) Given the claim exceeds 150, calculate the probability that it is less than or equal to 400.

Exercise 16.9 A company insures a group of 150 homogeneous and independent policyholders against an event that happens with probability 0.10. If the event happens, the amount of loss X has a uniform distribution on $(0, 1000]$. The company collects a premium of 55 from each policyholder. Estimate the probability that the total premium collected will not be sufficient to support the total loss.

16.7 Solutions to exercises

Exercise 16.1: In writing the expressions for the first two moments, rewrite the density function as

$$f_X(x) = \frac{\alpha}{\theta}\left(1+\frac{x}{\theta}\right)^{-\alpha-1}$$

and use the substitution $u = 1 + (x/\theta)$ in the integrals.

Exercise 16.2: From the given, we find $\alpha = 3.25$. This gives us 0.5320 for the answer.

Exercise 16.3: From (16.0.1), when $\alpha = 1$, we have $F_X(x) = 1-(\theta/(x+\theta))$. Use (16.0.7) to arrive at the following expression for the limited expected value: $\mathbb{E}[\min(X, d)] = \theta \ln(1 + \frac{d}{\theta})$.

Exercise 16.4: Observe that the insurer will pay only beyond the deductible d. So, it will pay no more than 5 if the loss does not reach $(d + 5)$. The given probability is therefore $\mathbb{P}(X \le d + 5|X > d) = 0.4092$. It can be shown that $F_X(x) = x^2/100$. This gives $d = 1.5$.

Exercise 16.5: $\mathbb{P}(X \le L) = 0.865$.

Exercise 16.6: The expected loss is 1000 for which insurer is responsible for 900. Therefore, the policyholder is responsible for 100, on the average.

Exercise 16.7: The expected reimbursement is $\mathbb{E}(X_I) = k\mu[e^{-d/\mu} - e^{-L/\mu}] = 0.90 * 800 * [e^{-25/800} - e^{-2000/800}] = 638.75$. To arrive at the desired d, we have $d = -\mu \cdot \log((575/(0.90 * 800) - e^{-2000/800}) = 101.63$.

Exercise 16.8: To calculate $\mathbb{P}(Y > 150)$, we can use direct application of (16.0.14) or use the law of total probability, i.e., $\mathbb{P}(Y > 150|I = 0)\mathbb{P}(I = 0) + \mathbb{P}(Y > 150|I = 1)\mathbb{P}(I = 1) = \mathbb{P}(Y > 150|I = 1)\mathbb{P}(I = 1)$. The first term vanishes because if event does not happen, then there is 0 insurance claim. Thus, using the property of Pareto, we have $\mathbb{P}(Y > 150) = 0.10 * (1 + (150/400))^{-2.5} = 0.451$. For part (b), the required probability is

$$\begin{aligned}
\mathbb{P}(Y \le 400|Y > 150) &= \frac{\mathbb{P}(150 \le Y < 400)}{\mathbb{P}(Y > 150)} = \frac{\mathbb{P}(Y > 150) - \mathbb{P}(Y > 400)}{\mathbb{P}(Y > 150)} \\
&= 1 - \frac{\mathbb{P}(Y > 400)}{\mathbb{P}(Y > 150)} = 1 - \frac{0.10 * (1 + (400/400))^{-2.5}}{0.10 * (1 + (150/400))^{-2.5}} = 0.608.
\end{aligned}$$

Exercise 16.9: The mean of the total losses is $\mathbb{E}(S_{150}) = 150 * 50 = 7500$ and the variance is $\mathrm{Var}(S_{150}) = 150 * 30833.33 = 4625000$. Total premium collected is $55 * 150 = 8250$. The required probability then is

$$\begin{aligned}
\mathbb{P}(S_{150} > 8250) &\approx \mathbb{P}(Z > (8250 - 7500)/\sqrt{4625000}) \\
&= \mathbb{P}(Z > 0.35) = 1 - \mathbb{P}(Z \le 0.35) = 1 - 0.63307 = 0.36693.
\end{aligned}$$