CHAPTER 17

* Applications of probability in finance

17.1. Coin toss games

17.1.1. The simple coin toss game. Suppose, as in Example 4.8, that we toss a fair coin repeatedly and independently. If it comes up heads, we win a dollar, and if it comes up tails, we lose a dollar. Unlike in Chapter 3 , we now can describe the solution using sums of independent random variables. We will use the partial sums process introduced in Definition 14.1 in Chapter 14

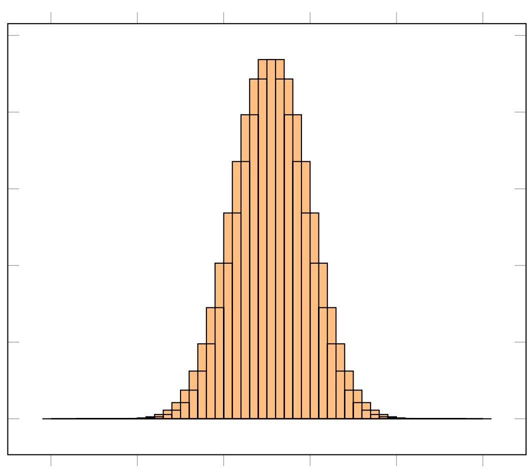
$$S_n = \sum_{i=1}^{n} X_i,$$

where $X_1, X_2, X_3, \ldots$ are i.i.d. random variables with the distribution $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = \frac{1}{2}$. Then S_n represents the total change in the number of dollars that we have after n coin tosses: if we started with $\$M$, we will have $M + S_n$ dollars after n tosses. The name *process* is used because the amount changes over time, and *partial sums* is used because we compute S_n before we know what is the final outcome of the game. The process S_n is also commonly called *the simple random walk*.

The Central Limit Theorem tells us that S_n is approximately distributed as a normal random variable with mean 0 and variance n, that is,

$$M_n = M + S_n \sim M + \sqrt{n} Z \sim \mathcal{N}(M, n)$$

and these random variables have the distribution function $F(x) = \Phi\left(\frac{x-M}{\sqrt{n}}\right)$.



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17.1.2. The coin toss game stopped at zero. Suppose the game is modified so that it is stopped when the amount of money reaches zero. Can we compute the probability distribution function of M_n, the amount of money after n coin tosses?

A useful trick, called the *Reflection Principle*, tells us that the probability to have x dollars after n coin tosses is

$$\mathbb{P}(M + S_n = x) - \mathbb{P}(M + S_n = -x) \quad \text{if } x > 0$$

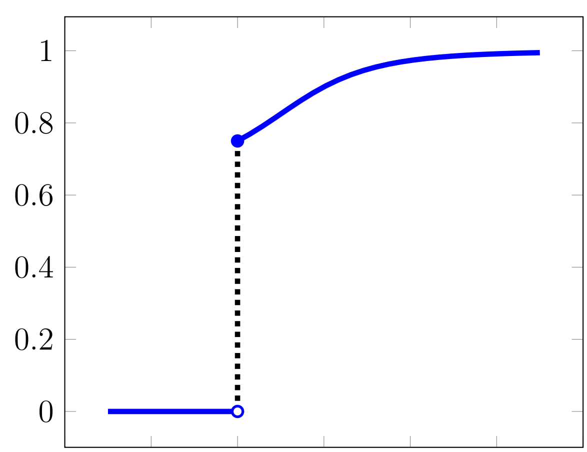
To derive this formula, we again denote by M_n the amount of money we have after n coin tosses. Then

$$\begin{aligned}
\mathbb{P}(M_n = x) &= \mathbb{P}(M + S_n = x, M + S_k > 0 \text{ for all } k = 1, 2, ..., n) \\
&= \mathbb{P}(M + S_n = x) - \mathbb{P}(M + S_n = x, M + S_k = 0 \text{ for some } k = 1, 2, ..., n) \\
&= \mathbb{P}(M + S_n = x) - \mathbb{P}(M + S_n = -x, M + S_k = 0 \text{ for some } k = 1, 2, ..., n) \\
&= \mathbb{P}(M + S_n = x) - \mathbb{P}(M + S_n = -x).
\end{aligned}$$

This, together with the Central Limit Theorem, implies that the cumulative probability distribution function of M_n can be approximated by

$$F(x) = \begin{cases} \Phi\left(\frac{x-M}{\sqrt{n}}\right) + \Phi\left(\frac{-x-M}{\sqrt{n}}\right) & \text{if } x \geqslant 0 \\ 0 & \text{otherwise.} \end{cases}$$

The following graph shows the approximate shape of this function.



Note that this function is discontinuous as the jump at zero represents the probability that we have lost all the money by the time n, that is,

$$\mathbb{P}(M_n = 0) \approx 2\Phi\left(-\frac{M}{\sqrt{n}}\right)$$

If we consider the limit $n \to \infty$, then $\mathbb{P}(M_n = 0) \xrightarrow[n\to\infty]{} 2\Phi(0) = 1$.

This proves that in this game all the money will be eventually lost with probability one. In fact, this conclusion is similar to the conclusion in Example 4.9.

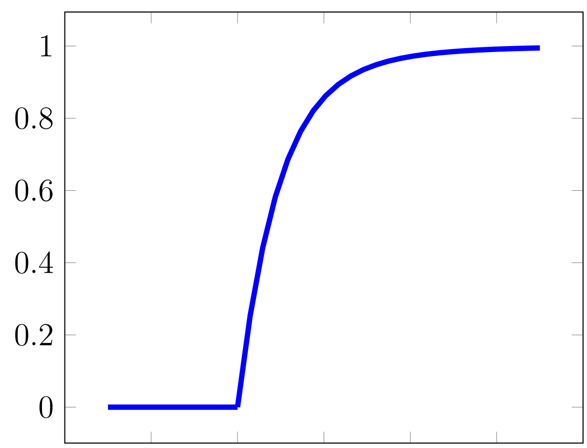
17.1.3. The coin toss game with borrowing at zero. Suppose now that the game is modified so that each time when we hit zero, instead of stopping, we borrow \$1 and continue playing. Another form of the *Reflection Principle* implies that the probability to have x dollars is

$$\mathbb{P}(M_n = x) = \mathbb{P}(M + S_n = x) + \mathbb{P}(M + S_n = -x) \quad \text{if } x > 0.$$

This formula is easy to explain because in this game the amount of money can be expressed as $M_n = |M + S_n|$. The Central Limit Theorem tells us that the cumulative probability distribution function of M_n can be approximated by

$$F(x) = \begin{cases} \Phi\left(\frac{x-M}{\sqrt{n}}\right) - \Phi\left(\frac{-x-M}{\sqrt{n}}\right) & \text{if } x \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

The following graph shows the approximate shape of this function.



17.1.4. Probability to win N dollars before reaching as low as L dollars. Continuing the simple coin toss game, suppose now that L, M and N are integers such that $L < M < N$. If we start with \$$M$, what is the probability that we will get up to \$$N$ before we go as low as \$$L$? As in Chapter 3, we are interested in finding the function

$$y(x) = \mathbb{P}\,(\text{ winning } \$N \text{ before reaching } \$L \mid M = x)$$

which satisfies $N - L + 1$ linear equations

$$y(x) = \begin{cases} 0 & \text{if } x = L \\ \ldots & \\ \frac{1}{2}(y(x+1) + y(x-1)) & \text{if } L < x < N \\ \ldots & \\ 1 & \text{if } x = N \end{cases}$$

In general, in more complicated games, such a function is called a *harmonic function* because its value at a given x is the average of the neighboring values. In our game we can compute that $y(x)$ is a linear function with slope $\frac{1}{N-L}$ which gives us the formula

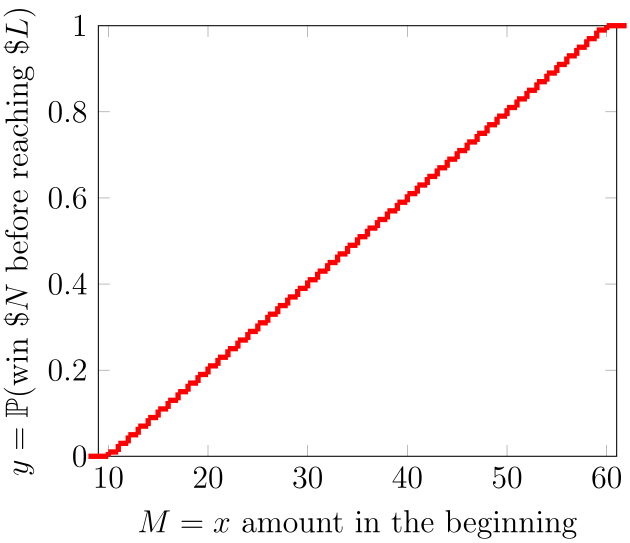
$$y(x) = \frac{x - L}{N - L}$$

and the final answer: with probability

$$\mathbb{P}(\text{ winning } \$N \text{ before reaching } \$L \mid M = x) = y(M) = \frac{\mathbf{M-L}}{\mathbf{N-L}} \tag{17.1.1}$$

we win $\$N$ before going as low as $\$L$ if we begin with $\$M$. Formula (17.1.1) applies in general to Gambler's Ruin problems, a particular case of which we consider in this section.

The following graph shows $y(x) = \frac{x-L}{N-L}$, the probability to win $\$N = \$60$ before reaching as low as $\$L = \$10$, in a game when $M_{n+1} = M_n \pm \$1$ with probability $1/2$ at each step.



17.1.5. Expected playing time. Suppose we play the same simple coin toss game as in the previous section, and we would like to compute the expected number of coin tosses needed to complete the game. If we denote this expected number by $T(x)$, we will have a system of $N - L + 1$ linear equations

$$\mathbb{E}T(x) = \begin{cases} 0 & \text{if } x = L \\ \dots & \\ 1 + \frac{1}{2}\left(\mathbb{E}T(x+1) + \mathbb{E}T(x-1)\right) & \text{if } L < x < N \\ \dots & \\ 0 & \text{if } x = N \end{cases}$$

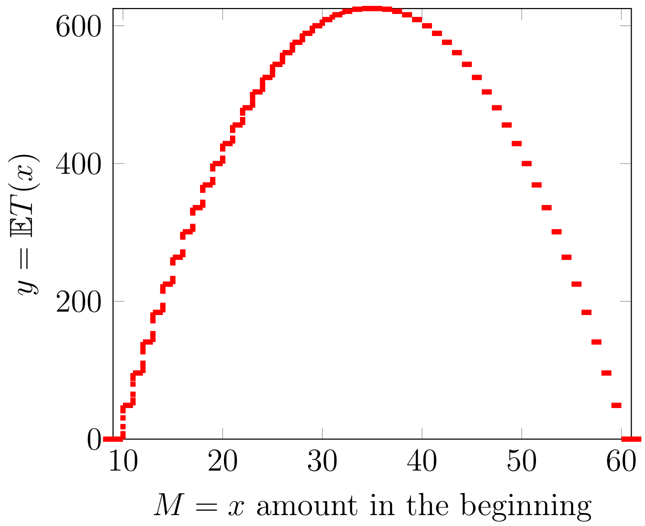
These equations have a unique solution given by the formula

$$\mathbb{E}T(x) = (x - L)(N - x)$$

and the final answer: the expected number of coin tosses is

$$\mathbb{E}\mathbf{T}(\mathbf{M}) = (\mathbf{M} - \mathbf{L})(\mathbf{N} - \mathbf{M}). \tag{17.1.2}$$

The following graph shows $\mathbb{E}T(x) = (x - L)(N - x)$, the expected number of coin tosses to win $\$N = \$60$ before reaching as low as $\$L = \$10$, in a game when $M_{n+1} = 2M_n$ or $M_{n+1} = \frac{1}{2}M_n$ with probability $1/2$ at each step.



17.1.6. Doubling the money coin toss game. Let us now consider a game in which we begin with $\$M$ dollars, toss a fair coin repeatedly and independently. If it comes up heads, we *double our money*, and if it comes up tails, we *lose half of our money*. If we start with $\$M$, what is the probability that we will get up to $\$N$ before we go as low as $\$L$?

To answer this question, we first should notice that our money M_n after n coin tosses is given as a partial product process $M_n = M \cdot Y_1 \cdot Y_2 \cdot \ldots \cdot Y_n$, where $Y_1, Y_2, Y_3, \ldots$ are i.i.d. random variables with the distribution $\mathbb{P}(Y_i = 2) = \mathbb{P}(Y_i = \frac{1}{2}) = \frac{1}{2}$. If again we write $y(x) = \mathbb{P}(\text{ winning } \$N \text{ before reaching } \$L)$, then

$$y(x) = \begin{cases} 0 & \text{if } x = L \\ \ldots & \\ \frac{1}{2}(y(2x) + y(\frac{1}{2}x)) & \text{if } L < x < N \\ \ldots & \\ 1 & \text{if } x = N \end{cases}$$

This function is linear if we change to the logarithmic variable $\log(x)$, which gives us the answer:

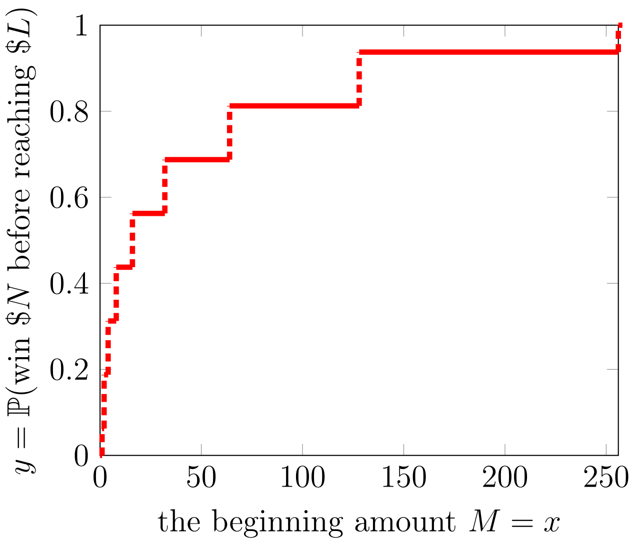
$$\mathbb{P}(\text{ winning } \$N \text{ before reaching } \$L) \approx \frac{\log(M/L)}{\log(N/L)}$$

This answer is approximate because, according to the rules, we can only have capital amounts represented by numbers $M2^k$, where k is an integer, and L, M, N maybe only approximately equal to such numbers. The exact answer is

$$\mathbb{P}(\text{ winning } \$N \text{ before reaching } \$L \mid M = x) = \frac{\mathbf{l}}{\mathbf{l} + \mathbf{w}}, \tag{17.1.3}$$

where l is the number of straight losses needed to reach $\$L$ from $\$M$ and w is the number of straight wins needed to reach $\$N$ from $\$M$. Equation (17.1.3) is again the general formula for Gambler's Ruin problems, the same as in Equation (17.1.1).

The following graph shows the probability to win $\$N = \$256$ before reaching as low as $\$L = \$1$ in a game when $M_{n+1} = 2M_n$ or $M_{n+1} = \frac{1}{2}M_n$ with probability $1/2$ at each step.



$y = \mathbb{P}$(win \$$N$ before reaching \$$L$)
1
0.8
0.6
0.4
0.2
0
0
50
100
150
200
250
the beginning amount $M = x$

17.2. Exercises on simple coin toss games

Exercise 17.1. In Subsection 17.1.1, what is the approximate distribution of $M_n - M_k$?

Exercise 17.2. In Subsection 17.1.1, find $\mathrm{Cov}(M_k, M_n)$.
Hint: assume $n > k$ and write $M_n = M_k + M_n - M_k = M_k + (S_n - S_k)$.

Exercise 17.3. Consider the game in which $M_n = Me^{\sigma S_n}$. Describe the rules of this game.

Exercise 17.4. In the game in Exercise 17.3, find $\mathbb{E}M_n$, $\mathbb{E}M_n^2$, $\mathrm{Var}(M_n)$.

Exercise 17.5. In the game in Exercise 17.3, how M_n and M_k are related?
Hint: assume $n > k$ and write $M_{n+1} = M_n \dfrac{M_{n+1}}{M_n}$. Also consider $M_n = M_k \dfrac{M_n}{M_k}$.

Exercise 17.6. Following Exercise 17.4, find $\mathrm{Cov}(M_n, M_k)$.

Exercise 17.7. In the game in Exercise 17.3, find the probability to win $\$N$ before reaching as low as $\$L$.

Exercise 17.8. In the game in Exercise 17.7, find the expected playing time.

Exercise 17.9. Following Exercise 17.3, use the normal approximation (as in the Central Limit Theorem) to find an approximate distribution of M_n. Then use this distribution to find approximate values of $\mathbb{E}M_n$, $\mathbb{E}M_n^2$, $\mathrm{Var}(M_n)$.

Exercise 17.10. Following Exercise 17.6, use the normal approximation (as in the Central Limit Theorem) to find the approximate value of $\mathrm{Cov}(M_n, M_k)$.

Exercise 17.11. Compare quantities in Exercises 17.4 and 17.9: which ones are larger and which ones are smaller? In which case a normal approximation gets better for larger n, and in which case it gets worse? If $n \to \infty$, how does σ need to behave in order to have an accurate normal approximation?

17.3. Problems motivated by the American options pricing

Problem 17.1. Consider the following game: a fair dice is thrown once and the player can either stop the game and receive the amount of money equals the outcome of the dice, or the player can decide to throw the dice the second time, and then receive the amount of money equals the outcome of the dice on this second throw. Compute the maximal expected value of the payoff and the corresponding optimal strategy.

Problem 17.2. Compute the maximal expected value of the payoff and the corresponding optimal strategy in the following game. A fair dice is thrown 3 times.

- After each throw except for the 3rd one, the player can either stop the game or continue.
- If the player decides to stop, then he/she receives the amount of money, which equals the current outcome of the dice (between 1 and 6).
- If the game is continued up to and including the 3rd throw, the player receives the amount of money, which equals to the outcome of the dice on the 3rd throw.

Problem 17.3.

(1) Compute the maximal expected value of the payoff and the corresponding optimal strategy in the same game as in Problem 17.2, but when up to 4, or 5, or 6 throws are allowed.
(2) Compute the maximal expected value of the payoff and the corresponding optimal strategy in the same game as in Problem 17.2, when an unlimited number of throws are allowed.

Problem 17.4. Let us consider a game where at each round, if you bet $\$x$, you get $\$2x$, if you win and \$0, if you lose. Let us also suppose that at each round, the probability of winning equals to the probability of losing and is equal to 1/2. Additionally, let us assume that the outcomes of every round are independent.

In such settings, let us consider the following *doubling strategy*. Starting from a bet of \$1 in the first round, you stop if you win or you bet twice as much if you lose. In such settings, if you win for the first (and only) time in the nth round, your cumulative winning is $\$2^n$. Show that

$$\mathbb{E}\left[\text{ cumulative winning }\right] = \infty.$$

This is called the *St. Petersburg paradox*. The paradox is in an observation that one would not pay an infinite amount to play such a game.

Notice that if the game is stopped at the nth round, the dollar amount you spent in the previous rounds is

$$2^0 + \cdots + 2^{n-2} = \left(2^0 + \cdots + 2^{n-2}\right)\frac{1-\frac{1}{2}}{1-\frac{1}{2}} = 2^{n-1} - 1.$$

Therefore, the dollar difference between the total amount won and the total amount spent is

$$2^{n-1} - (2^{n-1} - 1) = 1,$$

and *does not depend on* n. This seems to specify a riskless strategy of winning \$1. However, if one introduces a *credit constraint*, i.e., if a player can only spent $\$M$, for some fixed positive number M, then even if M is large, the expected winning becomes finite, and one cannot safely win \$1 anymore.

Problem 17.5. In the context of Problem 17.4, let G denotes the *cumulative winning*. Instead of computing the expectation of G, Daniel Bernoulli has proposed to compute the expectation of the *logarithm* of G. Show that

$$\mathbb{E}\left[\log_2(G)\right] = \log_2(g) < \infty$$

and find g.

Problem 17.6. Let us suppose that a random variable X, which corresponds to the dollar amount of winning in some lottery, has the following distribution

$$\mathbb{P}[X = n] = \frac{1}{Cn^2}, \quad n \in \mathbb{N},$$

where $C = \sum\limits_{n=1}^{\infty} \frac{1}{n^2}$, which in particular is finite. Clearly, X is finite-valued (with probability one). Show that nevertheless $\mathbb{E}[X] = \infty$.

As a historical remark, note that here $C = \zeta(2)$, where $\zeta(s) = \sum\limits_{k=1}^{\infty} \frac{1}{n^s}$ is the Riemann zeta function (or Euler-Riemann zeta function) of a complex variables s. It was first proven by Euler in 1735 that $\zeta(2) = \frac{\pi^2}{6}$.

Problem 17.7. Let us suppose that a one-year interest rate is determined at the beginning of each month. In this case $r_0, r_1, \ldots, r_{N-1}$ are such interest rates, where only r_0 is non-random. Thus \$1 of investment at time zero is worth $(1 + r_0)$ at the end of the year 1, $(1+r_0)(1+r_1)$ at the end of year 2, $(1+r_0)\ldots(1+r_{k-1})$ at the end of year k, and so forth. Let us suppose that $r_0 = 0.1$ and $(r_i)_{i=1,\ldots,N-1}$ are independent random variables with the following Bernoulli distribution (under so-called risk-neutral measure): $r_i = 0.15$ or $.05$ with probability $1/2$ each.

Compute the price at time 0 of the security that pays \$1 at time N. Note that such a security is called *zero-coupon bond*.

Hint: let D_N denotes the *discount factor*, i.e.,

$$D_N = \frac{1}{(1 + r_0)\ldots(1 + r_{N-1})}.$$

We need to evaluate

$$\widetilde{\mathbb{E}}\left[D_N\right].$$

Problem 17.8. In the settings of Problem 17.7, let *simply compounded yield* for the zero-coupon bond with maturity N is the number y, such that

$$\widetilde{\mathbb{E}}\left[D_N\right] = \frac{1}{(1+y)^m}.$$

Calculate y.

Problem 17.9. In the settings of Problem 17.7, let *continuously compounded yield* for the zero-coupon bond with maturity N is the number y, such that

$$\widetilde{\mathbb{E}}\left[D_N\right] = e^{-yN}.$$

Find y.

17.4. Problems on the Black-Scholes-Merton pricing formula

The following problems are related to the Black-Scholes-Merton pricing formula. Let us suppose that X is a standard normal random variable and

$$S(T) = S(0)\exp\left((r - \tfrac{1}{2}\sigma^2)T + \sigma\sqrt{T}X\right), \tag{17.4.1}$$

is the price of the stock at time T, where r is the interest rate, σ is the volatility, and $S(0)$ is the initial value. Here T, r, σ, and $S(0)$ are constants.

Problem 17.10. Show that

$$\mathbb{E}\left[e^{-rT}(S(T)-K)^+\right] = S(0)\Phi(d_1) - Ke^{-rT}\Phi(d_2), \tag{17.4.2}$$

where K is a positive constant,

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left(\log\left(\frac{S(0)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T\right), \quad d_2 = \frac{1}{\sigma\sqrt{T}}\left(\log\left(\frac{S(0)}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T\right),$$

and Φ is the cumulative distribution function of a standard normal random variable, i.e.

$$\Phi(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{1}{2}z^2}dz, \quad y \in \mathbb{R}.$$

Note that (17.4.2) the Black-Scholes-Merton formula, which gives the price of a European call option in at time 0 with strike K and maturity T.

Problem 17.11. In the framework of the Black-Scholes-Merton model, i.e., with the stock price process given by (17.4.1) with $r = 0$, let us consider

$$\mathbb{E}\left[S(t)^{1/3}\right]. \tag{17.4.3}$$

Find $\hat{t} \in [0,1]$ and evaluate $\mathbb{E}\left[S(\hat{t})^{1/3}\right]$ such that

$$\mathbb{E}\left[S(\hat{t})^{1/3}\right] = \max_{t\in[0,1]}\mathbb{E}\left[S(t)^{1/3}\right].$$

Note that $\max\limits_{t\in[0,1]}\mathbb{E}\left[S(t)^{1/3}\right]$ is closely related to the payoff of the *American cube root option* with maturity 1 and $\hat{t}$ to the optimal policy.

Problem 17.12. In the framework of the Black-Scholes-Merton model, i.e. with the stock price process given by (17.4.1), let us consider

$$\max_{t\in[0,1]}\mathbb{E}\left[e^{-rt}\left(S(t)-K\right)^+\right]. \tag{17.4.4}$$

Find $\hat{t} \in [0,1]$, such that

$$\mathbb{E}\left[e^{-r\hat{t}}\left(S(\hat{t})-K\right)^+\right] = \max_{t\in[0,1]}\mathbb{E}\left[e^{-rt}\left(S(t)-K\right)^+\right].$$

Similarly to Problem 17.11, $\max\limits_{t\in[0,1]}\mathbb{E}\left[e^{-rt}\left(S(t)-K\right)^+\right]$ is closely related to the payoff of the *American call option* with maturity 1 and $\hat{t}$ to the optimal policy.

17.5. Selected solutions

Answer to Exercise 17.4: The normal approximations imply

$$\begin{aligned}
\mathbb{E}M_n &= M\left(\frac{e^{\sigma}+e^{-\sigma}}{2}\right)^n, \\
\mathbb{E}M_n^2 &= M^2\left(\frac{e^{2\sigma}+e^{-2\sigma}}{2}\right)^n, \\
\mathrm{Var}(M_n) &= M^2\left(\left(\frac{e^{2\sigma}+e^{-2\sigma}}{2}\right)^n - \left(\frac{e^{\sigma}+e^{-\sigma}}{2}\right)^{2n}\right).
\end{aligned}$$

Answer to Exercise 17.9:

$$\begin{aligned}
\mathbb{E}M_n &\approx Me^{n\sigma^2/2}, \\
\mathbb{E}M_n^2 &\approx M^2e^{2n\sigma^2}, \\
\mathrm{Var}(M_n) &\approx M^2\left(e^{2n\sigma^2}-e^{n\sigma^2}\right).
\end{aligned}$$

Answer to Exercise 17.11:

$$\begin{aligned}
\mathbb{E}M_n &= M\left(\frac{e^{\sigma}+e^{-\sigma}}{2}\right)^n < Me^{n\sigma^2/2}, \\
\mathbb{E}M_n^2 &= M^2\left(\frac{e^{2\sigma}+e^{-2\sigma}}{2}\right)^n < M^2e^{2n\sigma^2}, \\
\mathrm{Var}(M_n) &= M^2\left(\left(\frac{e^{2\sigma}+e^{-2\sigma}}{2}\right)^n - \left(\frac{e^{\sigma}+e^{-\sigma}}{2}\right)^{2n}\right) < M^2\left(e^{2n\sigma^2}-e^{n\sigma^2}\right).
\end{aligned}$$

The normal approximations get better if σ is small, but get worse if n is large. The standard optimal regime is $n\to\infty$ and $n\sigma^2\to 1$, which means $\sigma\sim\frac{1}{\sqrt{n}}$.

Sketch of the solution to Problem 17.1: The strategy is to select a value x and say that the player stops if this value is exceeded after the first throw, and goes to the second through if this value is not exceeded. We know that the average value of one through is $(6+1)/2$ without any strategies. The probability to exceed x is $(6-x)/6$, and the conditional expectation of the payoff is $\frac{7+x}{2}$ if x is exceeded. So the expected payoff is $\frac{7+x}{2}\cdot\frac{6-x}{6}+\frac{7}{2}\cdot\frac{x}{6}$. This gives the optimal strategies for $x=3$ and the maximal expected payoff is $EP_2=4.25$.

Sketch of the solution to Problem 17.2: The expected payoff is $\frac{7+x}{2}\cdot\frac{6-x}{6}+\frac{17}{4}\cdot\frac{x}{6}$. Here $\frac{17}{4}=4.25$ replaces $\frac{7}{2}=3.5$ because after the first throw the player can decide either to stop, or play the game with two throws, which was solved and the maximal expected payoff was 4.25. So in the case of three throws, we have one optimal strategy with cut off $x_1=4$

after the first throw, and cut off $x_2 = 3$ after the second throw, following Problem 17.1. The expected payoff of the game which allows up to three throws is

$$\mathbb{E}P_3 = \frac{7+4}{2} \cdot \frac{6-4}{6} + \frac{17}{4} \cdot \frac{4}{6} = \frac{14}{3} \approx 4.6666$$

Answer to Problem 17.3:

(1) $\mathbb{E}P_4 = \frac{89}{18} \approx 4.9444$ $\qquad$ $\mathbb{E}P_5 = \frac{277}{54} \approx 5.1296$ $\qquad$ $\mathbb{E}P_6 = \frac{1709}{324} \approx 5.2747$

(2) $\mathbb{E}P_\infty = 6$.

Sketch of the solution to Problem 17.4: By direct computation, $\mathbb{E}\left[\textit{cumulative winning}\right]$ is given by the following divergent series

$$\sum_{k=1}^{\infty} 2^k \frac{1}{2^k} = \infty.$$

Sketch of the solution to Problem 17.5:

$$\mathbb{E}\left[\log_2(G)\right] = \sum_{k=1}^{\infty} \log_2(2^k) \frac{1}{2^k} = \sum_{k=1}^{\infty} \frac{k}{2^k} = 2 = \log_2(4) < \infty.$$

In particular, $g = 4$.

Sketch of the solution to Problem 17.6: $\mathbb{E}[X] = \frac{1}{C} \sum\limits_{n=1}^{\infty} \frac{1}{n} = \infty$, as the harmonic series is divergent.

Sketch of the solution to Problem 17.7: Using independence of r_ks, we get

$$\tilde{\mathbb{E}}\left[D_N\right] = \frac{1}{1+r_0} \prod_{k=1}^{N-1} \tilde{\mathbb{E}}\left[\frac{1}{1+r_k}\right] = \frac{1}{1+r_0} \left(\tilde{\mathbb{E}}\left[\frac{1}{1+r_1}\right]\right)^{N-1} = 0.909 \times 0.911^{N-1}.$$

Sketch of the solution to Problem 17.8: Direct computations give $y = \left(\frac{1}{\tilde{\mathbb{E}}[D_N]}\right)^{\frac{1}{N}} - 1$.

Sketch of the solution to Problem 17.9: Similarly to Problem 17.8, we get $y = -\frac{\log\left(\tilde{\mathbb{E}}[D_N]\right)}{N}$.

Sketch of the solution to Problem 17.10: From formula (17.4.1), we get

$$\mathbb{E}\left[e^{-rT}(S(T)-K)^+\right] = \int_{\mathbb{R}} e^{-rT} \max\left(S(0)e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}x} - K, 0\right) \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx.$$

Now, integration of the right-hand side yields the result.

Sketch of the solution to Problem 17.11: Let us fix $t \in [0,1]$. From Jensen's inequality (Proposition 15.7), we get $\mathbb{E}\left[S(t)^{1/3}\right] \leq \left(\mathbb{E}\left[S(t)\right]\right)^{1/3}$. The inequality is strict for $t > 0$, by strict concavity of $x \to x^{1/3}$. The equality is achieved at $t = 0$. Therefore $\hat{t} = 0$, and $\mathbb{E}\left[S(\hat{t})^{1/3}\right] = S(0)^{1/3}$.

Note that in the settings of this problem $\hat{t}$ is actually the optimal policy of the *American cube root option* and $\mathbb{E}\left[S(\hat{t})^{1/3}\right]$ is the corresponding price. However, in general one needs to consider $\max_{\tau\in[0,1]} \mathbb{E}\left[S(\tau)^{1/3}\right]$, where τ is what is known as *stopping times*, i.e. random times which an additional structural property.

Sketch of the solution to Problem 17.12: for every $t \in [0,1]$, we have

$$e^{-rt}\left(S(t)-K\right)^{+} = \left(S(t)e^{-rt} - Ke^{-rt}\right)^{+} \leqslant \left(S(t)e^{-rt} - Ke^{-r}\right)^{+}$$

and $\mathbb{E}\left[S(t)e^{-rt}\right] = S(0)$. Now, using convexity of $x \to (x-K)^{+}$ and applying Jensen's inequality (Proposition 15.7) for conditional expectation, we deduce that

$$\mathbb{E}\left[\left(S(t)e^{-rt} - Ke^{-r}\right)^{+}\right] < \mathbb{E}\left[\left(S(1)e^{-r} - Ke^{-r}\right)^{+}\right]$$

for every $t \in [0,1)$. We conclude that

$$\hat{t} = 1 \quad \text{and} \quad \mathbb{E}\left[e^{-r\hat{t}}\left(S(\hat{t}) - K\right)^{+}\right] = \mathbb{E}\left[e^{-r}\left(S(1)-K\right)^{+}\right].$$