CHAPTER 1

Combinatorics

1.1. Basic counting principle and combinatorics

1.1.1. Basic counting principle. The first basic counting principle is to multiply. Namely, if there are n possible outcomes of doing something and m outcomes of doing another thing, then there are $m \cdot n$ possible outcomes of performing both actions.

Basic counting principle

Suppose that two experiments are to be performed. Then if experiment 1 can result in any one of m possible outcomes, and if for each outcome of experiment 1 there are n possible outcomes of experiment 2, then there are $m \cdot n$ possible outcomes of the two experiments together.

Example 1.1. Suppose we have 4 shirts of 4 different colors and 3 pants of different colors. How many different outfits are there? For each shirt there are 3 different colors of pants, so altogether there are $4 \times 3 = 12$ possibilities.

Example 1.2. How many different license plate numbers with 3 letters followed by 3 numbers are possible?

The English alphabet has 26 different letters, therefore there are 26 possibilities for the first place, 26 for the second, 26 for the third, 10 for the fourth, 10 for the fifth, and 10 for the sixth. We multiply to get $(26)^3(10)^3$.

1.1.2. Permutations. How many ways can one arrange letters a, b, c? We can list all possibilities, namely,

$$abc \quad acb \quad bac \quad bca \quad cab \quad cba.$$

There are 3 possibilities for the first position. Once we have chosen the letter in the first position, there are 2 possibilities for the second position, and once we have chosen the first two letters, there is only 1 choice left for the third. So there are $3 \times 2 \times 1 = 6 = 3!$ arrangements. In general, if there are n distinct letters, there are $n!$ different arrangements of these letters.

Example 1.3. What is the number of possible batting orders (in baseball) with 9 players? Applying the formula for the number of permutations we get $9! = 362880$.

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Example 1.4. How many ways can one arrange 4 math books, 3 chemistry books, 2 physics books, and 1 biology book on a bookshelf so that all the math books are together, all the chemistry books are together, and all the physics books are together?

We can arrange the math books in 4! ways, the chemistry books in 3! ways, the physics books in 2! ways, and the biology book in $1! = 1$ way. But we also have to decide which set of books go on the left, which next, and so on. That is the same as the number of ways of arranging of four objects (such as the letters M, C, P, B), and there are 4! ways of doing that. We multiply to get the answer $4! \cdot (4! \cdot 3! \cdot 2! \cdot 1!) = 6912$.

In permutations the order **does** matter as is illustrated by the next example.

Example 1.5. How many ways can one arrange the letters a, a, b, c? Let us label them first as A, a, b, c. There are $4! = 24$ ways to arrange these letters. But we have repeats: we could have Aa or aA which are the same. So we have a repeat for each possibility, and so the answer should be $4!/2! = 12$.

If there were 3 as, 4 bs, and 2 cs, we would have

$$\frac{9!}{3!4!2!} = 1260.$$

What we just did is called finding the ***number of permutations***. These are permutations of a given set of objects (elements) unlike the example with the licence plate numbers where we could choose the same letter as many times as we wished.

Permutations

The number of permutations of n objects is equal to

$$n! := 1 \cdot ... \cdot n,$$

with the usual convention $0! = 1$.

1.1.3. Combinations. Now let us look at what are known as ***combinations***.

Example 1.6. How many ways can we choose 3 letters out of 5? If the letters are a, b, c, d, e and order matters, then there would be 5 choices for the first position, 4 for the second, and 3 for the third, for a total of $5 \times 4 \times 3$. Suppose now the letters selected were a, b, c. If order does not matter, in our counting we will have the letters a, b, c six times, because there are 3! ways of arranging three letters. The same is true for any choice of three letters. So we should have $5 \times 4 \times 3/3!$. We can rewrite this as

$$\frac{5 \cdot 4 \cdot 3}{3!} = \frac{5!}{3!2!} = 10$$

This is often written as $\binom{5}{3}$, read "5 *choose* 3". Sometimes this is written $C_{5,3}$ or ${}_5C_3$.

Combinations (binomial coefficients)

The number of different groups of k objects chosen from a total of n objects is equal to
$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

Note that this is true when the order of selection is irrelevant, and if the order of selection is relevant, then there are

$$n \cdot (n-1) \cdot \ldots \cdot (n-k+1) = \frac{n!}{(n-k)!}$$

ways of choosing k objects out of n.

Example 1.7. How many ways can one choose a committee of 3 out of 10 people? Applying the formula of the number of combinations we get $\binom{10}{3} = 120$.

Example 1.8. Suppose there are 8 men and 8 women. How many ways can we choose a committee that has 2 men and 2 women? We can choose 2 men in $\binom{8}{2}$ ways and 2 women in $\binom{8}{2}$ ways. The number of possible committees is then the product

$$\binom{8}{2} \cdot \binom{8}{2} = 28 \cdot 28 = 784.$$

Example 1.9. Suppose one has 9 people and one wants to divide them into one committee of 3, one committee of 4, and the last one of 2. There are $\binom{9}{3}$ ways of choosing the first committee. Once that is done, there are 6 people left and there are $\binom{6}{4}$ ways of choosing the second committee. Once that is done, the remainder must go in the third committee. So the answer is

$$\frac{9!}{3!6!}\frac{6!}{4!2!} = \frac{9!}{3!4!2!}.$$

Example 1.10. For any $k \leqslant n$ we have that

choosing k objects is the same as rejecting $n-k$ objects,

$$\binom{n}{k} = \binom{n}{n-k}.$$

Indeed, the left-hand side gives the number of different groups of k objects chosen from a total of n objects which is the same to choose $n-k$ objects not to be in the group of k objects which is the number on the right-hand side.

Combinations (multinomial coefficients)

The number of ways to divide n objects into one group of n_1 objects, one group of n_2, ..., and a rth group of n_r objects, where $n = n_1 + \cdots + n_r$, is equal to

$$\binom{n}{n_1, ..., n_r} = \frac{n!}{n_1!n_2!\cdots n_r!}.$$

Example 1.11. Suppose we have 4 Americans and 6 Canadians.

(a) How many ways can we arrange them in a line?

(b) How many ways if all the Americans have to stand together?

(c) How many ways if not all the Americans are together?

(d) Suppose you want to choose a committee of 3, which will be all Americans or all Canadians. How many ways can this be done?

(e) How many ways for a committee of 3 that is not all Americans or all Canadians?

For (a) we can simply use the number of arrangements of 10 elements, that is, 10!.

For (b) we can consider the Americans as one group (element) and each Canadian as a distinct group (6 elements); this gives 7 distinct groups (elements) to be arranged, which can be done in 7! ways. Once we have these seven groups arranged, we can arrange the Americans within their group in 4! ways, so we get 4!7! by the basic counting principle.

In (c) the answer is the answer to (a) minus the answer to (b): $10! - 4!7!$

For (d) we can choose a committee of 3 Americans in $\binom{4}{3}$ ways and a committee of 3 Canadians in $\binom{6}{3}$ ways, so the answer is $\binom{4}{3} + \binom{6}{3}$.

Finally for (e) we can choose a committee of 3 out of 10 in $\binom{10}{3}$ ways, so the answer is $\binom{10}{3} - \binom{4}{3} - \binom{6}{3}$.

Finally, we consider three interrelated examples.

Example 1.12. First, suppose one has 8 copies of o and two copies of $|$. How many ways can one arrange these symbols in order? There are 10 spots, and we want to select 8 of them in which we place the os. So we have $\binom{10}{8}$.

Example 1.13. Next, suppose one has 8 indistinguishable balls. How many ways can one put them in 3 boxes? Let us use sequences of os and $|$ s to represent an arrangement of balls in these 3 boxes; any such sequence that has $|$ at each side, 2 other $|$ s, and 8 os represents a way of arranging balls into boxes. For example, if one has

$$|\quad o\quad o\quad |\quad o\quad o\quad o\quad |\quad o\quad o\quad o\quad |,$$

this would represent 2 balls in the first box, 3 in the second, and 3 in the third. Altogether there are $8 + 4$ symbols, the first is a $|$ as is the last, so there are 10 symbols that can be either $|$ or o. Also, 8 of them must be o. How many ways out of 10 spaces can one pick 8 of them into which to put a o? We just did that, so the answer is $\binom{10}{8}$.

Example 1.14. Now, to finish, suppose we have \$8,000 to invest in 3 mutual funds. Each mutual fund required you to make investments in increments of \$1,000. How many ways can we do this? This is the same as putting 8 indistinguishable balls in 3 boxes, and we know the answer is $\binom{10}{8}$.

1.2. Further examples and explanations

1.2.1. Generalized counting principle. Here we expand on the basic counting principle formulated in Section 1.1.1. One can visualize this principle by using the box method below. Suppose we have two experiments to be performed, namely, one experiment can result in n outcomes, and the second experiment can result in m outcomes. Each box represents the number of possible outcomes in that experiment.

$$\boxed{\text{Experiment 1}} \qquad \boxed{\text{Experiment 2}} = \boxed{\text{Experiment 1 and 2 together}}$$

$$\boxed{\mathbf{m}} \qquad \boxed{\mathbf{n}} = \boxed{\mathbf{mn}}$$

Example 1.15. There are 20 teachers and 100 students in a school. How many ways can we pick a teacher and student of the year? Using the box method we get $20 \times 100 = 2000$.

Generalized counting principle

Suppose that k experiments are to be performed, with the number of possible outcomes being n_i for the ith experiment. Then there are

$$n_1 \cdot \ldots \cdot n_k$$

possible outcomes of all k experiments together.

Example 1.16. A college planning committee consists of 3 freshmen, 4 sophomores, 5 juniors, and 2 seniors. A subcommittee of 4 consists of 1 person from each class. How many choices are possible? The counting principle or the box method gives $3 \times 4 \times 5 \times 2 = 120$.

Example 1.17 (Example 1.2 revisited)**.** Recall that for 6-place license plates, with the first three places occupied by letters and the last three by numbers, we have $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10$ choices. What if *no* repetition is allowed? We can use the counting principle or the box method to get $26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8$.

Example 1.18. How many functions defined on k points are possible if each function can take values as either 0 or 1. The counting principle or the box method on the $1, \ldots, k$ points gives us 2^k possible functions. This is the generalized counting principle with $n_1 = n_2 = \ldots = n_k = 2$.

1.2.2. Permutations. Now we give more examples on permutations, and we start with a general results on the number of possible permutations, see also Combinations (multinomial coefficients) on page 6.

Permutations revisited

The number of different permutations of n objects of which n_1 are alike, n_2 are alike, ..., n_r are alike is equal to
$$\frac{n!}{n_1! \cdots n_r!}.$$

Example 1.19. How many ways can one arrange 5 math books, 6 chemistry books, 7 physics books, and 8 biology books on a bookshelf so that all the math books are together, all the chemistry books are together, and all the physics books are together.

We can arrange the math books in 5! ways, the chemistry in 6! ways, the physics in 7! ways, and biology books in 8! ways. We also have to decide which set of books go on the left, which next, and so on. That is the same as the number of ways of arranging the letters M,C,P, and B, and there are 4! ways of doing that. So the total is $4! \cdot (5! \cdot 6! \cdot 7! \cdot 8!)$ ways.

Now consider a couple of examples with *repetitions*.

Example 1.20. How many ways can one arrange the letters a, a, b, b, c, c?

Let us first re-label the letters by A, a, B, b, C, c. Then there are $6! = 720$ ways to arrange these letters. But we have repeats (for example, Aa or aA) which produce the same arrangement for the original letters. So dividing by the number of repeats for A, a, B, b and C, c, so the answer is

$$\frac{6!}{(2!)^3} = 90.$$

Example 1.21. How many different letter arrangements can be formed from the word PEPPER?

There are three copies of P and two copies of E, and one of R. So the answer is

$$\frac{6!}{3!2!1!} = 60.$$

Example 1.22. Suppose there are 4 Czech tennis players, 4 U.S. players, and 3 Russian players, in how many ways could they be arranged, if we do not distinguish players from the same country? By the formula above we get $\frac{11!}{4!4!3!}$.

1.2.3. Combinations.
Below are more examples on combinations.

Example 1.23. Suppose there are 9 men and 8 women. How many ways can we choose a committee that has 2 men and 3 women?

We can choose 2 men in $\binom{9}{2}$ ways and 3 women in $\binom{8}{3}$ ways. The number of committees is then the product

$$\binom{9}{2} \cdot \binom{8}{3}.$$

Example 1.24. Suppose somebody has n friends, of whom k are to be invited to a meeting.

(1) How many choices do exist for such a meeting if two of the friends will not attend together?
(2) How many choices do exist if two of the friends will only attend together?

We use a similar reasoning for both questions.

(1) We can divide all possible groups into two (disjoint) parts: one is for groups of friends none of which are these two, and another which includes exactly one of these two friends. There are $\binom{n-2}{k}$ groups in the first part, and $\binom{n-2}{k-1}$ in the second. For the latter we also need to account for a choice of one out of these two incompatible friends. So altogether we have

$$\binom{n-2}{k} + \binom{2}{1} \cdot \binom{n-2}{k-1}$$

(2) Again, we split all possible groups into two parts: one for groups which have none of the two inseparable friends, and the other for groups which include both of these two friends. Then

$$\binom{n-2}{k} + 1 \cdot 1 \cdot \binom{n-2}{k-2}.$$

Theorem 1.1 The binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

PROOF. We give two proofs.

First proof: let us expand the left-hand side $(x+y) \cdot \ldots \cdot (x+y)$. This is the sum of 2^n terms, and each term has n factors. For now we keep each product in the order we expanded the left-hand side, therefore we have all possible (finite) sequences of variables x and y, with the total power being n. We would like to collect all the terms having the same number of xs and ys.

Counting all the terms having k copies of x and $n-k$ copies of n is the same as asking in a sequence of n positions, how many ways can one choose k of them in which to put x. The answer is $\binom{n}{k}$ which gives the coefficient for $x^k y^{n-k}$. To illustrate it we take $k=2$ and $n=3$, then all possible terms are

$$x \cdot x \cdot y \qquad x \cdot y \cdot x \qquad y \cdot x \cdot x$$

Second proof: we will use (mathematical) induction on n. For $n = 1$ we have that the left-hand side is $x + y$, and the right-hand side

$$\sum_{k=0}^{1} \binom{1}{k} x^k y^{1-k} = \binom{1}{0} x^0 y^{1-0} + \binom{1}{1} x^1 y^{1-1}$$
$$= y + x = x + y,$$

so the statement holds for $n = 1$. Suppose now that the statement holds for $n = N$, we would like to show it for $n = N + 1$.

$$(x+y)^{N+1} = (x+y)(x+y)^N = (x+y)\sum_{k=0}^{N} \binom{N}{k} x^k y^{N-k}$$
$$= x\sum_{k=0}^{N} \binom{N}{k} x^k y^{N-k} + y\sum_{k=0}^{N} \binom{N}{k} x^k y^{N-k}$$
$$= \sum_{k=0}^{N} \binom{N}{k} x^{k+1} y^{N-k} + \sum_{k=0}^{N} \binom{N}{k} x^k y^{N-k+1}$$
$$= \sum_{k=1}^{N+1} \binom{N}{k-1} x^k y^{N-k+1} + \sum_{k=0}^{N} \binom{N}{k} x^k y^{N-k+1},$$

where we replaced k by $k - 1$ in the first sum. Then we see that

$$(x+y)^{N+1} = \binom{N}{N} x^{N+1} y^0 + \sum_{k=1}^{N} \left(\binom{N}{k-1} + \binom{N}{k} \right) x^k y^{N-k+1} + \binom{N}{0} x^0 y^{N+1}$$
$$= x^{N+1} + \sum_{k=1}^{N} \left(\binom{N}{k-1} + \binom{N}{k} \right) x^k y^{N-k+1} + y^{N+1} = \sum_{k=0}^{N+1} \binom{N+1}{k}.$$

Here we used Example 1.26. □

Example 1.25. We can use combinatorics to show that

$$\tbinom{10}{4} = \tbinom{9}{3} + \tbinom{9}{4}$$

without evaluating these expressions explicitly.

Indeed, the left-hand side represents the number of committees consisting of 4 people out of the group of 10 people. Now we would like to represent the right-hand side. Let's say Tom Brady is one these ten people, and he might be in one of these committees and he is special, so we want to know when he will be there or not. When he is in the committee of 4, then there are $1 \cdot \binom{9}{3}$ number of ways of having a committee with Tom Brady as a member, while $\binom{9}{4}$ is the number of committees that do not have Tom Brady as a member. Adding it up gives us the number of committees of 4 people chosen out of the 10.

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$

$$\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7}$$

Pascal's triangle

Example 1.26. More generally we have

the Pascal's triangle inductive rule

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

which can be proven for $k = 1, 2, ..., n-1$ either using the same argument as in Example 1.25 or a formula for binomial coefficients.

Example 1.27. Expand $(x+y)^3$. This can be done by applying Theorem 1.1, The binomial theorem, to get $(x+y)^3 = y^3 + 3xy^2 + 3x^2y + x^3$.

1.2.4. Multinomial coefficients.

Example 1.28. Suppose we are to assign 10 police officers: 6 patrols, 2 in station, 2 in schools. Then there are $\frac{10!}{6!2!2!}$ different assignments.

Example 1.29. We have 10 flags: 5 of them are blue, 3 are red, and 2 are yellow. These flags are indistinguishable, except for their color. Then there are $\frac{10!}{5!3!2!}$ many different ways we can order them on a flag pole.

Example 1.30 (Example 1.13 revisited)**.** Suppose one has n indistinguishable balls. How many ways can one put them in k boxes, assuming $n > k$?

As in Example 1.13 we use sequences of os and $|$ s to represent each an arrangement of balls in boxes; any such sequence that has $|$ at each side, $k-1$ copies of $|$ s, and n copies of os. How many different ways can we arrange this, if we have to start with $|$ and end with $|$? Between these, we are only arranging $n+k-1$ symbols, of which only n are os. So the question can be re-formulated as this: how many ways out of $n+k-1$ *spaces* can one pick n of *them* into which to put an o? This gives $\binom{n+k-1}{n}$. Note that this counts all possible ways including the ones when some of the boxes can be empty.

Suppose now we want to distribute n balls in k boxes so that none of the boxes are empty. Then we can line up n balls represented by os, instead of putting them in boxes we can place $|$ s in spaces between them. Note that we should have a $|$ on each side, as all balls have to be put to a box. So we are left with $k-1$ copies of $|$ s to be placed among n balls. This means that we have $n-1$ places, and we need to pick $k-1$ out of these to place $|$ s. So we can reformulate the problem as choose $k-1$ places out of $n-1$, and so the answer is $\binom{n-1}{k-1}$.

We can check that for $n=3$ and $k=2$ we indeed have 4 ways of distributing three balls in two boxes, and only two ways if every box has to have at least one ball.

1.3. Exercises

Exercise 1.1. Suppose a license plate must consist of 7 numbers or letters. How many license plates are there if

(A) there can only be letters?
(B) the first three places are numbers and the last four are letters?
(C) the first three places are numbers and the last four are letters, but there can not be any repetitions in the same license plate?

Exercise 1.2. A school of 50 students has awards for the top math, English, history and science student in the school

(A) How many ways can these awards be given if each student can only win one award?
(B) How many ways can these awards be given if students can win multiple awards?

Exercise 1.3. A password can be made up of any 4 digit combination.

(A) How many different passwords are possible?
(B) How many are possible if all the digits are odd?
(C) How many can be made in which all digits are different or all digits are the same?

Exercise 1.4. There is a school class of 25 people made up of 11 guys and 14 girls.

(A) How many ways are there to make a committee of 5 people?
(B) How many ways are there to pick a committee of all girls?
(C) How many ways are there to pick a committee of 3 girls and 2 guys?

Exercise 1.5. If a student council contains 10 people, how many ways are there to elect a president, a vice president, and a 3 person prom committee from the group of 10 students?

Exercise 1.6. Suppose you are organizing your textbooks on a book shelf. You have three chemistry books, 5 math books, 2 history books and 3 English books.

(A) How many ways can you order the textbooks if you must have math books first, English books second, chemistry third, and history fourth?
(B) How many ways can you order the books if each subject must be ordered together?

Exercise 1.7. If you buy a Powerball lottery ticket, you can choose 5 numbers between 1 and 59 (picked on white balls) and one number between 1 and 35 (picked on a red ball). How many ways can you

(A) win the jackpot (guess all the numbers correctly)?
(B) match all the white balls but not the red ball?
(C) match exactly 3 white balls and the red ball?
(D) match at least 3 white balls and the red ball?

Exercise 1.8. A couple wants to invite their friends to be in their wedding party. The groom has 8 possible groomsmen and the bride has 11 possible bridesmaids. The wedding party will consist of 5 groomsmen and 5 bridesmaids.

(A) How many wedding party's are possible?
(B) Suppose that two of the possible groomsmen are feuding and will only accept an invitation if the other one is not going. How many wedding parties are possible?
(C) Suppose that two of the possible bridesmaids are feuding and will only accept an invitation if the other one is not going. How many wedding parties are possible?
(D) Suppose that one possible groomsman and one possible bridesmaid refuse to serve together. How many wedding parties are possible?

Exercise 1.9. There are 52 cards in a standard deck of playing cards. The poker hand consists of five cards. How many poker hands are there?

Exercise 1.10. There are 30 people in a communications class. Each student must interview one another for a class project. How many total interviews will there be?

Exercise 1.11. Suppose a college basketball tournament consists of 64 teams playing head to head in a knockout style tournament. There are 6 rounds, the round of 64, round of 32, round of 16, round of 8, the final four teams, and the finals. Suppose you are filling out a bracket, such as this, which specifies which teams will win each game in each round.

How many possible brackets can you make?

Exercise 1.12. We need to choose a group of 3 women and 3 men out of 5 women and 6 men. In how many ways can we do it if 2 of the men refuse to be chosen together?

Exercise 1.13. Find the coefficient in front of x^4 in the expansion of $(2x^2 + 3y)^4$.

Exercise 1.14. In how many ways can you choose 2 or less (maybe none!) toppings for your ice-cream sundae if 6 different toppings are available? (You can use combinations here, but you do not have to. Next, try to find a general formula to compute in how many ways you can choose k or less toppings if n different toppings are available

Exercise* 1.1. Use the binomial theorem to show that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n,$$
$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

Exercise* 1.2. Prove the multinomial theorem

$$(x_1 + ... + x_k)^n = \sum_{\substack{(n_1, \cdots, n_k) \\ n_1 + \cdots + n_k = n}} \binom{n}{n_1, ..., n_k} x_1^{n_1} \cdot ... \cdot x_k^{n_k}.$$

Exercise* 1.3. Show that there are $\binom{n-1}{k-1}$ distinct positive integer-valued vectors $(x_1, ..., x_k)$ satisfying

$$x_1 + ... + x_k = n, x_i > 0 \text{ for all } i = 1, ..., k.$$

Exercise* 1.4. Show that there are $\binom{n+k-1}{k-1}$ distinct non-positive integer-valued vectors $(x_1, ..., x_k)$ satisfying

$$x_1 + ... + x_k = n, x_i \geqslant 0 \text{ for all } i = 1, ..., k.$$

Exercise* 1.5. Consider a smooth function of n variables. How many different partial derivatives of order k does f possess?

1.4. Selected solutions

Solution to Exercise 1.1(A): in each of the seven places we can put any of the 26 letters giving

$$26^7$$

possible letter combinations.

Solution to Exercise 1.1(B): in each of the first three places we can place any of the 10 digits, and in each of the last four places we can put any of the 26 letters giving a total of

$$10^3 \cdot 26^4.$$

Solution to Exercise 1.1(C): if we can not repeat a letter or a number on a license plate, then the number of license plates becomes

$$(10 \cdot 9 \cdot 8) \cdot (\cdot 26 \cdot 25 \cdot 24 \cdot 23)\,.$$

Solution to Exercise 1.2(A):

$$50 \cdot 49 \cdot 48 \cdot 47$$

Solution to Exercise 1.2(B):

$$50^4$$

Solution to Exercise 1.3(A):

$$10^4$$

Solution to Exercise 1.3(B):

$$5^4$$

Solution to Exercise 1.3(C):

$$10 \cdot 9 \cdot 8 \cdot 7 + 10$$

Solution to Exercise 1.4(A):

$$\binom{25}{5}$$

Solution to Exercise 1.4(B):

$$\binom{14}{5}$$

Solution to Exercise 1.4(C):

$$\binom{14}{3} \cdot \binom{11}{2}$$

Solution to Exercise 1.5:

$$10 \cdot 9 \cdot \binom{8}{3}$$

Solution to Exercise 1.6(A):

$$5!3!3!2!$$

Solution to Exercise 1.6(B):

$$4!\,(5!3!3!2!)$$

Solution to Exercise 1.7(A):

$$1$$

Solution to Exercise 1.7(B):

$$1 \cdot 34$$

Solution to Exercise 1.7(C):

$$\binom{5}{3} \cdot \binom{54}{2} \cdot \binom{1}{1}$$

Solution to Exercise 1.7(D):

$$\binom{5}{3} \cdot \binom{54}{2} \cdot \binom{1}{1} + \binom{5}{4} \cdot \binom{54}{1} \cdot \binom{1}{1} + 1$$

Solution to Exercise 1.8(A):

$$\binom{8}{5} \cdot \binom{11}{5}$$

Solution to Exercise 1.8(B):

$$\binom{6}{5} \cdot \binom{11}{5} + \binom{2}{1} \cdot \binom{6}{4} \cdot \binom{11}{5}$$

Solution to Exercise 1.8(C):

$$\binom{8}{5} \cdot \binom{9}{5} + \binom{8}{5} \cdot \binom{2}{1} \cdot \binom{9}{4}$$

Solution to Exercise 1.8(D):

$$\binom{7}{5} \cdot \binom{10}{5} + 1 \cdot \binom{7}{4} \cdot \binom{10}{5} + \binom{7}{5} \cdot 1 \cdot \binom{10}{4}$$

Solution to Exercise 1.9:

$$\binom{52}{5}$$

Solution to Exercise 1.10:

$$\binom{30}{2}$$

Solution to Exercise 1.11: First notice that the 64 teams play 63 total games: 32 games in the first round, 16 in the second round, 8 in the 3rd round, 4 in the regional finals, 2 in the final four, and then the national championship game. That is, $32+16+8+4+2+1=$ 63. Since there are 63 games to be played, and you have two choices at each stage in your bracket, there are 2^{63} different ways to fill out the bracket. That is,

$$2^{63} = 9{,}223{,}372{,}036{,}854{,}775{,}808.$$

Solution to Exercise* 1.1: use the binomial formula

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

with $x = y = 1$ to see

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot 1^k \cdot 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k},$$

and with $x = -1$, $y = 1$

$$0 = (-1+1)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot (-1)^k \cdot (1)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k .$$

Solution to Exercise* 1.2: we can prove the statement using mathematical induction on k. For $k = 1$ we have

$$(x_1)^n = \sum_{n_1=n} \binom{n}{n_1} x_1 = x_1^n,$$

which is true; for $k = 2$ we have

$$(x_1 + x_2)^n = \sum_{\substack{(n_1,n_2)\\ n_1+n_2=n}} \binom{n}{n_1, n_2} x_1^{n_1} \cdot x_2^{n_2} = \sum_{n_1=0}^{n} \binom{n}{n_1} x_1^{n_1} \cdot x_2^{n-n_1},$$

which is the binomial formula itself. Now suppose the multinomial formula holds for $k = K$ (*induction hypothesis*), that is,

$$(x_1 + ... + x_K)^n = \sum_{\substack{(n_1,\cdots,n_K)\\ n_1+\cdots+n_K=n}} \binom{n}{n_1,...,n_K} \cdot x_1^{n_1} \cdot ... \cdot x_K^{n_K},$$

and we need to show

$$(x_1 + ... + x_{K+1})^n = \sum_{\substack{(n_1,\cdots,n_{K+1})\\ n_1+\cdots+n_{K+1}=n}} \binom{n}{n_1,...,n_{K+1}} \cdot x_1^{n_1} \cdot ... \cdot x_{K+1}^{n_{K+1}}.$$

Denote

$$y_1 = x_1, .., y_{K-1} := x_{K-1}, y_K := x_K + x_{K+1},$$

then by the induction hypothesis

$$(x_1 + ... + x_{K+1})^n = (y_1 + ... + y_K)^n = \sum_{\substack{(n_1,\cdots,n_K)\\ n_1+\cdots+n_K=n}} \binom{n}{n_1,...,n_K} \cdot y_1^{n_1} \cdot ... \cdot y_{K-1}^{n_{K-1}} \cdot y_K^{n_K}$$

$$= \sum_{\substack{(n_1,\cdots,n_K)\\ n_1+\cdots+n_K=n}} \binom{n}{n_1,...,n_K} \cdot x_1^{n_1} \cdot ... \cdot x_{K-1}^{n_{K-1}} \cdot (x_K + x_{K+1})^{n_K} .$$

By the binomial formula

$$(x_K + x_{K+1})^{n_K} = \sum_{m=1}^{n_K} \binom{n_K}{m} \cdot x_K^m \cdot x_{K+1}^{n_K-m},$$

therefore

$$(x_1+...+x_{K+1})^n = \sum_{\substack{(n_1,\cdots,n_K)\\ n_1+\cdots+n_K=n}} \binom{n}{n_1,...,n_K} \cdot x_1^{n_1}\cdot ... \cdot x_{K-1}^{n_{K-1}} \cdot \sum_{m=1}^{n_K} \binom{n_K}{m} \cdot x_K^m \cdot x_{K+1}^{n_K-m}.$$

It is easy to see (using the definition of multinomial coefficients) that

$$\binom{n}{n_1,...,n_K}\binom{n_K}{m} = \binom{n}{n_1,...,n_K,m}, n_1+...+n_K+m=n.$$

Indeed,

$$\begin{aligned}\binom{n}{n_1,...,n_K}\binom{n_K}{m} &= \frac{n!}{n_1!n_2!\cdot ... \cdot n_{K-1}!\cdot n_K!}\frac{n_K!}{m!\,(n_K-m)!}\\ &= \frac{n!}{n_1!n_2!\cdot ... \cdot n_{K-1}!\cdot m!\,(n_K-m)!} = \binom{n}{n_1,...,n_K,m}.\end{aligned}$$

Thus

$$(x_1+...+x_{K+1})^n = \sum_{\substack{(n_1,\cdots,n_K)\\ n_1+\cdots+n_K=n}} \sum_{m=1}^{n_K} \binom{n}{n_1,...,n_K,m} \cdot x_1^{n_1}\cdot ... \cdot x_{K-1}^{n_{K-1}} \cdot \cdot x_K^m \cdot x_{K+1}^{n_K-m}.$$

Note that $n_K = m+(n_K-m)$, so if we denote $m_1 := n_1, m_2 := n_2, ..., m_{K-1} := n_{K-1}, m_K := m, m_{K+1} := n_K - m$ then we see that

$$(x_1+...+x_{K+1})^n = \sum_{\substack{(m_1,\cdots,m_K,m_{K+1})\\ m_1+\cdots+m_{K+1}=n}} \binom{n}{m_1,...,m_K,m_{K+1}} \cdot x_1^{m_1}\cdot ... \cdot x_{K-1}^{m_{K-1}} \cdot x_K^{m_K} \cdot x_{K+1}^{m_{K+1}}$$

which is what we wanted to show.

Solution to Exercise* 1.3: this is the same problem as dividing n indistinguishable balls into k boxes in such a way that each box has at least one ball. To do so, you can select $k-1$ of the $n-1$ spaces between the objects. There are $\binom{n-1}{k-1}$ possible selections that is equal to the number of possible positive integer solutions to the equation.

Solution to Exercise* 1.4: define $y_i := x_i + 1$ and apply the previous problem.

Solution to Exercise* 1.5: the same answer as in the previous problem.